UNIFORM K-HOMOLOGY THEORY
— DRAFT —

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ABSTRACT. We define a uniform version of analytic K-homology theory for separable, proper metric spaces. Furthermore, we define an index map from this theory into K-theory of uniform Roe C*-algebras, analogous to the coarse assembly map from analytic K-homology into the K-theory of Roe C*-algebras (cf. [11, 12, 15]). We show that our theory has a Mayer–Vietoris sequence. We prove that for a torsion-free countable discrete group Γ, the direct limit of the uniform K-homology of the Rips complexes of Γ, $\lim_{d\to\infty} K_*^u(P_d\Gamma)$, is isomorphic to $K_*^{\mathrm{top}}(\Gamma,\ell^\infty\Gamma)$, the left-hand side of the Baum–Connes conjecture with coefficients in $\ell^\infty\Gamma$ (cf. [14]). In particular, this provides a computation of uniform K-homology groups for some torsion-free groups. As an application of uniform K-homology, we prove a criterion for amenability in terms of vanishing of a "fundamental class", in spirit of similar criteria in uniformly finite homology [3] and K-theory of uniform Roe algebras [6].

1. INTRODUCTION

The analytic K-homology theory of a second countable locally compact Hausdorff topological space X (see e.g. [8]) can be understood as an attempt to organize the elliptic differential operators over the space X into an abelian group. The (higher) indices of these operators can be interpreted as K-theory elements over C^*X, the Roe C*-algebra [12]. The Coarse Baum–Connes and coarse Novikov conjectures assert certain properties of this index (or coarse assembly) map $\mu : K_*(X) \to K_*(C^*X)$, and have applications in geometry (see e.g. [12, 15]. Also, the Coarse Baum–Connes conjecture can be viewed as an algorithm to compute the K-theory of Roe C*-algebras. In this spirit, the work presented here is setting up a framework for obtaining an algorithm to compute the K-theory groups of uniform Roe C*-algebras.

In this paper, we define a refined version of analytic K-homology theory. We loosely follow the exposition [8] of analytic K-homology. The main idea is to quantify "how well approximable by finite rank operators" are various compact operators appearing in the definition of a Fredholm module.

Our theory, compared to the classical K-homology, has some advantages (the theory becomes sensitive to some coarse properties, for instance amenability), but also some disadvantages (the K-theory of uniform Roe algebras tends to be uncountable if nonzero).

The theory exhibits similarities to the uniformly finite homology theory of Block and Weinberger [3, 4], which should be connected to our theory via a Chern character map. This is analogous to the Chern map from analytic K-homology into the locally finite homology groups.

Using estimates from [11], we show that some elliptic operators coming from geometry give rise to uniform K-homology classes. Furthermore, we construct an index map μ_{u} from uniform K-homology into the K-theory of uniform Roe C*-algebras. The original example of a coarse index theorem [11] is actually carried out in this uniform context.

We prove that amenability of a metric space is equivalent to non-vanishing of a "fundamental class" in the uniform K-homology of the space. Our criterion is parallel to similar criteria in the uniformly finite homology [3] and K-theory of uniform Roe algebras [6]. Our proof borrows ideas from both of these papers.

In the case when the space in question is a Cayley graph of a countable torsion–free group Γ, we show that $\lim_{d\to\infty} K_*^u(P_d\Gamma)$, the direct limit of uniform K-homologies of its Rips complexes is naturally isomorphic to $K_*^{\mathrm{top}}(\Gamma,\ell^\infty\Gamma)$, the left–hand side of the Baum–Connes conjecture for the group Γ with coefficients in $\ell^\infty(\Gamma)$. This is analogous to a result of Yu [14], where he shows the equivalence of the Coarse Baum–Connes conjecture and the Baum–Connes conjecture with coefficients in $\ell^\infty(\Gamma,\mathcal{K})$. This statement is true without any assumption on torsion; it is open whether the torsion-free assumption can be dropped in the uniform setting. On the other hand, since the Baum–Connes conjecture with commutative coefficients is known for a number of torsion–free groups, this result provides a computation of some uniform K-homology groups.

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The structure of this paper is as follows: Section 2 introduces the uniform K-homology groups, and in section 3 we prove that certain Dirac-type differential operators give rise to uniform K-homology classes. Sections 4 and 5 are devoted to proving the Mayer–Vietoris sequence in our theory. We turn to coarse geometry and the index map in sections 7–9. The connection between the uniform K-homology of a group Γ and the Baum–Connes conjecture with coefficients in $\ell^\infty\Gamma$ is shown in section 10. In the final section 11, we prove our criterion for amenability.

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2. UNIFORM K-HOMOLOGY GROUPS

In this paper, the spaces are separable proper metric spaces, unless explicitly specified otherwise. Throughout the paper, X shall stand for such a space, and d will denote its metric. Finally, to avoid set-theoretic difficulties, all Hilbert spaces are assumed to be separable.

Recall the definition of Fredholm modules — the representatives of cycles in the classical analytic K-homology theory.

Definition 2.1 (Fredholm modules). Let (H,ϕ,S) be a triple consisting of a Hilbert space H, a *-homomorphism $\phi : C_0(X) \to \mathscr{B}(H)$ and an operator $S \in \mathscr{B}(H)$. We say that such a triple is a 0-*Fredholm module* (or *even* Fredholm module), if for every $f \in C_0(X)$ the following hold:

- (Fredholmness) $(1-S^*S)\phi(f) \in \mathscr{K}(H)$ and $(1-SS^*)\phi(f) \in \mathscr{K}(H)$,
- (pseudolocality) $[S,\phi(f)] \in \mathscr{K}(H)$.

Similarly, we say that a triple (H,ϕ,P) as above is a 1-*Fredholm module* (or *odd* Fredholm module), if for every $f \in C_0(X)$:

- $(P^2-1)\phi(f) \in \mathscr{K}(H)$ and $(P-P^*)\phi(f) \in \mathscr{K}(H)$,
- (pseudolocality) $[P,\phi(f)] \in \mathscr{K}(H)$.

Remark 2.2. We can also formulate the Fredholmness condition for even Fredholm modules in another form, which is more convenient for the setting of differential operators: A triple (H,ϕ,T) forms an even Fredholm module, if H is $\mathbb{Z}_2$-graded, $\phi(f)$ is of degree 0 (i.e. even) for all $f \in C_0(X)$ and $T \in \mathscr{B}(H)$ is a pseudolocal operator of degree 1 (odd), satisfying that $(T^2-1)\phi(f) \in \mathscr{K}(H)$ and $(T^*-T)\phi(f) \in \mathscr{K}(H)$ for all $f \in C_0(X)$.

We modify this concept, defining "uniform Fredholm modules", which shall represent elements in the uniform K-homology theory. We introduce uniformity by "quantifying" the compactness of an operator in the following way: given $\varepsilon > 0$ we try to approximate our compact operator within ε by a finite rank operator with the smallest possible rank. In the definition of a Fredholm module, instead of just one compact operator, we really have a collection of compacts, depending on $f \in C_0(X)$, and we require a uniform bound on the ranks of ε–approximants for fixed R — a "scale" in the metric of X. This consideration is sufficient to ensure uniformity on the large scale. However, we want (certain) first order differential operators to give rise to uniform K-homology classes. The approximation properties of compacts arising from the pseudolocality condition really depend not only on the support of f but also on its derivative (just consider an operator $[D,f]$), and so we need to build in also some local control.

Specifically, for a metric space X and $R,L \geq 0$ we denote

$$\begin{aligned} C_R(X) &= \{f \in C_c(X) \mid \operatorname{diam}(\operatorname{supp}(f)) \leq R \text{ and } \|f\|_\infty \leq 1\} \\ C_{R,L}(X) &= \{f \in C_R(X) \mid f \text{ is } L\text{-continuous}\}. \end{aligned}$$

We say that $f : X \to Y$ is L-continuous, if there exists a nondecreasing function $\alpha : [0,\infty] \to [0,\infty)$ with $\alpha'(0) \geq \frac{1}{L}$, such that for any $x,y \in X$ we have $d(x,y) \leq \alpha(s) \implies d(f(x),f(y)) \leq s$. Loosely, one could formulate the condition as "locally L-Lipschitz". In particular, if a function is L-Lipschitz, then it is L-continuous (with $\alpha(s) = \frac{1}{L}s$). The converse is true for instance when X is a geodesic space. Hence for practical purposes we can replace this condition with just L-Lipschitz. We use the notion of L-continuity to emphasize the local side of being Lipschitz.

The reason for introducing L-continuity is the following: if X is a manifold and $f \in C_{R,L}(X)$ is differentiable at $x \in X$, then the norm of the derivative df of f at x is at most L. This observation is used in a crucial way in section 3, when proving that Dirac–type operators produce uniform K-homology classes. If one doesn't require the theory to include such classes, it is possible to just ignore L's and l-'s throughout the paper.

Furthermore $\bigcup_{L\geq 0} C_{R,L}(X)$ is dense in $C_R(X)$. This is completely analogous to saying that (once) differentiable functions are dense in the space of all continuous functions. The proof is outlined at the end of this section, in Lemma 2.18.

In the following definition, we introduce the uniformity conditions. We list two versions — one without the "L–dependency" and one featuring L.

Definition 2.3 (Uniform approximability)**.** Let H be a Hilbert space, X a metric space and $\phi : C_0(X) \to \mathscr{B}(H)$ a *-homomorphism. For $\varepsilon, M > 0$, an operator $T \in \mathscr{B}(H)$ is said to be (ε, M)*–approximable*, if there is a rank-M operator k, such that $||T - k|| < \varepsilon$.

Let $E(\cdot)$ (or $E(f)$) stand for an expression with operators in $\mathscr{B}(H)$ and terms $\phi(\cdot)$ (or $\phi(f)$). (For instance $E(\cdot) = T\phi(\cdot)$ or $E(f) = [T, \phi(f)]$.)

- For $\varepsilon, M, R > 0$, an expression $E(\cdot)$ is said to be $(\varepsilon, R, M; \phi)$*–approximable*, if for each $f \in C_R(X)$, $E(f)$ is (ε, M)–approximable.
- For $\varepsilon, R, L, M > 0$, an expression $E(\cdot)$ is said to be $(\varepsilon, R, L, M; \phi)$*–approximable*, if for each $f \in C_{R,L}(X)$, $E(f)$ is (ε, M)–approximable.
- An expression $E(\cdot)$ is *uniformly approximable*, if for every $R \geq 0, \varepsilon > 0$ there exists $M > 0$, such that $E(\cdot)$ is $(\varepsilon, R, M; \phi)$–approximable. Furthermore, we write $E_1(\cdot) \sim_{ua} E_2(\cdot)$, if the difference $E_1(\cdot) - E_2(\cdot)$ is uniformly approximable.
- An expression $E(\cdot)$ is *l-uniformly approximable*, if for every $R, L \geq 0, \varepsilon > 0$ there exists $M > 0$, such that $E(\cdot)$ is $(\varepsilon, R, L, M; \phi)$–approximable. Furthermore, we write $E_1(\cdot) \sim_{lua} E_2(\cdot)$, if the difference $E_1(\cdot) - E_2(\cdot)$ is l-uniformly approximable.

We introduce a special cases of uniform approximability:

- We say that an operator $T \in \mathscr{B}(H)$ is *uniform*, if $T\phi(\cdot)$ and $\phi(\cdot)T$ are uniformly approximable (i.e. $T\phi(f) \sim_{ua} 0 \sim_{ua} \phi(f)T$). We also say that T is $(\varepsilon, R, M; \phi)$*–uniform*, if both operators $\phi(f)T$, $T\phi(f)$ are $(\varepsilon, R, M; \phi)$–approximable.
- An operator $T \in \mathscr{B}(H)$ is said to be *uniformly pseudolocal*, if $[T, \phi(\cdot)]$ is uniformly approximable (i.e. $[T, \phi(f)] \sim_{ua} 0$).
- An operator $T \in \mathscr{B}(H)$ is said to be *l-uniformly pseudolocal*, if $[T, \phi(\cdot)]$ is l-uniformly approximable (i.e. $[T, \phi(f)] \sim_{lua} 0$).

Remark 2.4. The property of being uniformly pseudolocal is obviously stronger than that of being l-uniformly pseudolocal. In the former, we can obtain a bound M on ranks of approximants, which is independent of L (local condition), and depends only on R (support condition) and of course on ε.

Remark 2.5. The notion of an "l-uniform" operator is in fact equivalent to the notion of a uniform operator given above. More precisely, if $T\phi(\cdot)$ and $\phi(\cdot)T$ are l-uniformly approximable, then they are in fact just uniformly approximable, i.e. we can get a bound on M independent of L. In other words, checking uniformity of operator on "nice" function is sufficient. Indeed, for every $f \in C_R(X)$ we can construct a function $\tilde{f} \in C_{R+1,1}(X)$, such that $f\tilde{f} = f$. Now given $R, \varepsilon > 0$, if M is the constant such that $T\phi(\cdot)$ and $\phi(\cdot)T$ are $(\varepsilon, R+1, 1, M; \phi)$–approximable, then $\phi(f)T = \phi(f)\phi(\tilde{f})T$ and $T\phi(f) = T\phi(\tilde{f})\phi(f)$ are $(\varepsilon, R, M; \phi)$–approximable. Such an $\tilde{f}$ can be constructed for instance as $\tilde{f}(x) = \max(0, 1 - d(x, \mathrm{supp}(f)))$. One easily checks that this function is 1–Lipschitz, and so $\tilde{f} \in C_{\mathrm{diam}(\mathrm{supp}(f))+1,1}(X)$.

In the view of the previous remark, we can completely disregard the constant L appearing in the definition above, when we work with uniform operators only. This is so for instance in sections 6–11.

Definition 2.6 (Uniform Fredholm modules)**.** Let (H, ϕ, S) be a 0-Fredholm module. It is said to be *uniform*, if S is l-uniformly pseudolocal and the operators $1 - SS^*$, $1 - S^*S$ are uniform.

Let (H, ϕ, Q) be a 1-Fredholm module. It is said to be *uniform*, if Q is l-uniformly pseudolocal and the operators $1 - Q^2$ and $Q - Q^*$ are uniform.

Remark 2.7. By using "uniform Fredholm module" (without 0- or 1-) in a statement we shall mean that the statement applies to both 0- and 1- uniform Fredholm modules.

Remark 2.8. If we are given a Hilbert space H together with a *-homomorphism $\phi : C_0(X) \to \mathscr{B}(H)$ (i.e. an action of $C_0(X)$ on H), we say that (H,ϕ), or just H, is an X-*module*. When no confusion about ϕ can arise, we identify $f \in C_0(X)$ with $\phi(f) \in \mathscr{B}(H)$. Similarly, we omit "$\phi$" from $(\varepsilon, R, M; \phi)$, etc.

Example 2.9 (Fundamental class). Let Y be a uniformly discrete space. Let S be the unilateral shift operator on $\ell^2\mathbb{N}$ (i.e. a Fredholm operator with index 1). Denote $H = \ell^2 Y \otimes \ell^2\mathbb{N}$, and set $\tilde{S} = \mathrm{diag}(S) \in \mathscr{B}(H)$. Endow H with the multiplication action of $C_0(Y)$. More precisely, define $\phi : C_0(Y) \to \mathscr{B}(H)$ by $\phi(f)(\zeta(y)) = f(y)\zeta(y)$, for $\zeta : Y \to \ell^2\mathbb{N}$, a square summable function, and $y \in Y$, $f \in C_0(Y)$.

It is easy to check that $(H,\phi,\tilde{S})$ is a 0-uniform Fredholm module for Y ($\tilde{S}$ is actually uniformly pseudolocal). This module has pivotal role in our characterization of amenability in Section 11.

The following example is concerned with the K-homology classes coming from elliptic differential operators on manifolds.

Example 2.10. Let M be a complete Riemannian manifold and S a smooth complex vector bundle over M. Let D be a symmetric elliptic differential operator operating on sections of S. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a chopping function (an odd smooth function, $\chi(t) > 0$ for $t > 0$, $\chi(t) \to \pm 1$ for $t \to \pm\infty$.

Denote $H = L^2(M,S)$ and let $\rho : C_0(M) \to \mathscr{B}(H)$ be the multiplication action. It is proved in [8, Section 10.8], that $(H,\rho,\chi(D))$ is a Fredholm module (whether odd or even depends on the dimension of M).

Assuming that M has bounded geometry and that the operator D is "geometric" (e.g. has finite propagation speed), this Fredholm module is actually uniform. The proof is outlined in section 3.

We now proceed towards the definition of uniform K-homology groups.

Definition 2.11 (Homotopy)**.** A collection (H,ϕ_t,S_t), $t \in [0,1]$, of uniform Fredholm modules is a *homotopy*, if:

- $t \mapsto S_t$ is continuous in norm,
- the C*-algebras $\Theta(\phi_t) \subset \mathscr{B}(H)$ generated by all the uniform operators with finite propagation[1] all the same for all $t \in [0,1]$.

By an *operator homotopy* we mean a homotopy as above, with the restriction that $\phi_t = \phi_0$ for all $t \in [0,1]$.

Remark 2.12. The second condition above is satisfied if for instance ϕ_t's fulfill

- there exists $R > 0$, such that for $f,g \in C_0(X)$ with $d(\mathrm{supp}(f),\mathrm{supp}(g)) \geq R$, we have $\phi_s(f)\phi_t(g) = 0$ for all $s,t \in [0,1]$,
- for every $s,t \in [0,1]$ and $R > 0$, there are R' and M, so that every $\phi_t(f)$, $f \in C_R(X)$, is within a rank-M operator from one of the form $\phi_s(g)$, $g \in C_{R'}(X)$.

We now proceed as in [8, section 8.2] in defining a K-homology theory.

Given two uniform Fredholm modules, we can clearly form their direct sum, which becomes again a uniform Fredholm module.

Definition 2.13 (K_*^u)**.** We define the uniform K-homology group $K_i^u(X)$, $i = 0,1$, to be an abelian group generated by the unitary equivalence classes of uniform i-Fredholm modules (H,ϕ,S) with the following relations:

- if two uniform Fredholm modules $\mathbf{x},\mathbf{y}$ are homotopic, we declare $[\mathbf{x}] = [\mathbf{y}]$,
- for two uniform Fredholm modules $\mathbf{x},\mathbf{y}$, we set $[\mathbf{x} \oplus \mathbf{y}] = [\mathbf{x}] + [\mathbf{y}]$.

Recall that a Fredholm module (H,ϕ,S) is called *degenerate*, if the conditions in the definition hold exactly, that is $(1 - S^*S) = (1 - SS^*) = [\phi(f),S] = 0$ for all $f \in C_0(X)$ for the 0- version; and $S - S^* = S^2 - 1 = [\phi(f),S] = 0$ for all $f \in C_0(X)$ for the 1- version. The K_*^u-class of a degenerate Fredholm module is 0: the proof of the analogous result for K-homology [8, 8.2.8]) carries over verbatim.

The additive inverse of $[(H,\phi,S)] \in K_0^u(X)$ is $[(H,\phi,-S^*)]$. Similarly, the additive inverse of $[(H,\phi,P)] \in K_1^u(X)$ is $[(H,\phi,-P)]$. Again, the proof of these facts is just as [8, proof of 8.2.10]. For instance, $\left(\begin{smallmatrix} \cos tS & \sin tI \\ \sin tI & -\cos tS^* \end{smallmatrix}\right)$, $t \in [0,\frac{\pi}{2}]$, is a homotopy showing that $[(H,\phi,S)] + [(H,\phi,-S^*)] = [(H \oplus H, \phi \oplus \phi, \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right))] = 0 \in K_0^u(X)$.

It follows from the facts in the last two paragraphs, that every element of $K_*^u(X)$ can be represented as a class of a single uniform Fredholm module. Furthermore, $[\mathbf{x}] = [\mathbf{y}]$ in $K_*^u(X)$ if and only if there exists

[1] Recall (see 6.4) that $T \in \mathscr{B}(H)$ has *finite propagation*, if there exists $R \geq 0$, such that for any $f,g \in C_0(X)$ whose supports are at least R apart, we have $\phi_t(f)T\phi_t(g) = 0$.

a degenerate Fredholm module $\mathbf{z}$, such that $\mathbf{x}\oplus\mathbf{z}$ and $\mathbf{y}\oplus\mathbf{z}$ are unitarily equivalent to a pair of homotopic uniform Fredholm modules. In this case, we say that $\mathbf{x}$ and $\mathbf{y}$ are *stably homotopic*. Therefore, we may reformulate the definition of $K^u_*(X)$ as follows:

Proposition 2.14. *The group $K^u_i(X)$ is canonically isomorphic to the semigroup of stable homotopy equivalence classes of uniform i-Fredholm modules.*

The uniform K-homology is not functorial under continuous maps in general; we need two extra condition in order to obtain functoriality: one handling the large–scale and one taking care of the local phenomena.

Definition 2.15 (Uniform coboundedness; see definition 6.1)**.** A (not necessarily continuous) map $g: X\to Z$ between metric spaces X and Z is said to be *uniformly cobounded*, if for any $r\geq 0$, we have

$$R_g(r) := \sup_{z\in Z}\operatorname{diam}(g^{-1}(B(z,r))) < \infty.$$

Observe that an L-continuous uniformly cobounded map $g: X\to Z$ descends to a homomorphism on the uniform K-homology groups $g_*: K^u_*(X)\to K^u_*(Z)$ by the following observation: Take a uniform Fredholm module $(H,\phi: C_0(X)\to\mathscr{B}(H),S)$ of an $K^u_*(X)$-element. We denote by $\tilde g: C_0(Z)\to C_0(X)$ the induced *-homomorphism. Then there is a *-homomorphism $\phi\circ\tilde g: C_0(Z)\to\mathscr{B}(H)$. By uniform coboundedness, we obtain that if $f\in C_R(Z)$, then $\tilde g(f)\in C_{R_g(R)}(X)$. By L-continuity, $f\in C_{R,L'}(Z)$ implies $\tilde g(f)\in C_{R_g(R),LL'}(X)$. Hence the uniformity requirements transfer and $(H,\phi\circ\tilde g,S)$ becomes a uniform Fredholm module representing a $K^u_*(Z)$-element. We define $g_*[(H,\phi,S)] = [(H,\phi\circ\tilde g,S)]$.

We now prove a simple lemma analogous to a similar statement in the classical K-homology:

Lemma 2.16 (Compact perturbations)**.** *If (H,ϕ,T) is a uniform Fredholm module and $K\in\mathscr{B}(H)$ is uniform, then (H,ϕ,T) and $(H,\phi,T+K)$ are operator homotopic.*

Proof. We need to show that $(H,\phi,T+tK)$, $t\in[0,1]$ are uniform Fredholm modules. Fix $\varepsilon,R,L>0$ and let M be such that all the operators K, $[T,\phi(f)]$, $(1-T^*T)\phi(f)$ and $(1-TT^*)\phi(f)$ (or $(1-T^2)\phi(f)$ and $(T-T^*)\phi(f)$ in the 1-case) are (ε,M)—approximable for $f\in C_{R,L}(X)$.

First, for $f\in C_{R,L}(X)$, we have that $[T+tK,\phi(f)] = [T,\phi(f)] + tK\phi(f) - t\phi(f)K$, which is clearly $(3\varepsilon,3M)$–approximable. Hence the pseudolocality requirement is satisfied.

Let us now deal with the 0-case. Examine the following expression: $1-(T+tK)(T+tK)^* = (1-TT^*) - tKT^* - tTK^* - t^2KK^*$. Taking $f\in C_{R,L}(X)$ and multiplying by $\phi(f)$ the previous formula on the right, each of the elements $(1-TT^*)\phi(f)$, $tTK^*\phi(f)$, $t^2KK^*\phi(f)$ is going to be $(||T||||K||\varepsilon,M)$–approximable by assumption. We can rewrite the remaining term as follows $tKT^*\phi(f) = tK\phi(f)T^* + tK[T^*,\phi(f)]$, and so it is $(2||T||||K||\varepsilon,R,2M)$–approximable. Therefore, $(1-(T+tK)(T+tK)^*)\phi(f)$ is $(5||T||||K||\varepsilon,5M)$–approximable. It is clear that similar considerations can be applied to $1-(T+tK)^*(T+tK)$ as well.

Finally, we consider the 1-case. We see that $((T+tK)-(T^*+tK^*))\phi(f)$ is $(2\varepsilon,2M)$–approximable. Furthermore for $f\in C_{R,L}(X)$, $(1-(T+tK)^2)\phi(f) = (1-T^2)\phi(f) - tTK\phi(f) - t^2K^2\phi(f) - tK\phi(f)T - tK[T,\phi(f)]$, which is $(5||T||||K||\varepsilon,5M)$–approximable. □

As a first application of the previous lemma, we make the following observation:

Remark 2.17. We may always assume that a K^u_1-element is represented by a uniform 1-Fredholm module (H,ϕ,Q) with Q selfadjoint. It is because if we take any Q, $\frac{1}{2}(Q+Q^*)$ is selfadjoint and $Q-\frac{1}{2}(Q+Q^*) = \frac{1}{2}(Q-Q^*)$ is uniform. Moreover, the procedure of replacing Q by a selfadjoint operator can be applied to whole homotopies.

We finish the section by a lemma promised earlier.

Lemma 2.18. *Let X be a metric space. Given any compactly supported continuous function $f: X\to\mathbb{C}$ and $\varepsilon>0$, there exists $L>0$ and an L-continuous function $g: X\to\mathbb{C}$, such that $||f-g||_\infty<\varepsilon$.*

Proof. Without loss of generality we can assume that $f(X)\subset[0,1]$. Take an integer N, such that $\frac{1}{N}<\varepsilon$, and set $U_n = f^{-1}[0,\frac{n}{N}]$, $n=0,\dots,N$. Then $U_0\subset U_1\subset\cdots\subset U_N = X$ are closed sets. By uniform continuity, there exists $\delta>0$, such that $d(x,y)<\delta$ implies $|f(x)-f(y)|<\frac{1}{N}$. This implies that $N_\delta(U_n)\subset U_{n+1}$.

Define $g : X \to \mathbb{R}$ as follows: for $x \in X$, let $n(x)$ be such that $x \in U_{n(x)}$, but $x \notin U_{n(x)-1}$ (where we set $U_{-1} = \emptyset$). Now set $g(x) = \frac{n(x)-1}{N} + \frac{1}{N} \cdot \min(1, \frac{1}{\delta} d(x, U_{n(x)-1}))$ if $n(x) > 0$, and $g(x) = 0$ if $n(x) = 0$. It is clear from the construction that $\|f - g\|_\infty \leq \frac{1}{N} < \varepsilon$ and it is easy to verify that g is $\frac{1}{N\delta} < \frac{\varepsilon}{\delta}$-continuous. □

3. DIRAC–TYPE OPERATORS

In this section, we outline the proof of the fact that "geometric" operators on complete Riemannian manifolds with bounded geometry give rise to uniform Fredholm modules. The hard work was already done in [11], where it is shown that such geometric operators have index defined in the algebraic K-theory of the algebra $\mathscr{U}_{-\infty}(M)$ of operators given by smooth uniformly bounded kernels, the precursor of the uniform Roe algebra.

Recall the setting: Let M be a complete Riemannian manifold (without boundary) and S a Clifford bundle over M. More precisely, denote by Cliff(M) the complexified bundle of Clifford algebras Cliff(T_xM) (equipped with a natural connection), and let S be a smooth complex vector bundle over M equipped with an action of Cliff(M) and a compatible connection. The bundle S is graded, if in addition it is equipped with an involution ϵ anticommuting with the Clifford action of tangent vectors (see Remark 2.2).

A "geometric" operator will be a first-order differential operator D defined by the composition

$$\Gamma(S) \to \Gamma(T^*M \otimes S) \to \Gamma(TM \otimes S) \to \Gamma(S),$$

where the arrows are given by the connection, metric and Clifford multiplication, respectively. In local coordinates, this operator has the form

$$D = \sum e_k \frac{\partial}{\partial x_k}.$$

The signature and Dirac operators are of this type. The main properties of these operators is that they are elliptic, and have finite propagation (in a sense that there exists a constant C, such that supp$(e^{itD}\xi) \subset N_{Ct}(\text{supp}(\xi))$ for all $\xi \in \Gamma_c(S)$)[2].

Denote $H = L^2(S)$ and let $\rho : C_0(M) \to \mathscr{B}(H)$ be the multiplication action. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a chopping function (an odd smooth function, $\chi(t) > 0$ for $t > 0$, $\chi(t) \to \pm 1$ for $t \to \pm\infty$). Then $(H, \rho, \chi(D))$ is a Fredholm module (see [8, sections 10.6 and 10.8]). This is true in more general context, namely for any first-order elliptic differential operator on a complex smooth vector bundle. However to obtain uniformity, bounded geometry assumption and some analysis from [11] is required.

Following [11, section 2], we say that a Riemannian manifold M has *bounded geometry*, if it has positive injectivity radius and the curvature tensor is uniformly bounded, as are all its covariant derivatives. A bundle S has bounded geometry, if its curvature tensor, as well as all its covariant derivatives, are uniformly bounded. By [11, Proposition 2.4], bounded geometry can be seen by existence of nice coordinate patches, such that the Christoffel symbols comprise a bounded set in the Fréchet structure on C^∞.

For the record, let us collect all the assumptions and the conclusion into a theorem.

Theorem 3.1. *Let D be a geometric operator (as described above) on a Clifford bundle S over a complete Riemannian manifold with bounded geometry. For any chopping function χ, the triple $(L^2(S), \rho, \chi(D))$ is a uniform Fredholm module.*

The idea of the proof (which will be made more precise afterward) is as follows: it is proved in [11, theorem 5.5], that if $\varphi \in C_0(\mathbb{R})$ satisfies $\varphi^{(k)}(t) \leq C_k(1+|t|)^{m-k}$, then $\varphi(D)$ extends to a continuous map between Sobolev spaces $W^r \to W^{r-m}$ for any r. Now a bounded piece of our manifold can be transferred to a torus. The Fourier coefficients of a W^{-k}–function on a torus decay faster than $s \mapsto \frac{1}{s^k}$. Hence the finite rank approximants to the inclusion $W^{r-m} \hookrightarrow W^r$ can be constructed just by truncating the Fourier series — and knowing the rate of decay of the coefficients tells us how big rank do we need for a given $\varepsilon > 0$ — independently on the position of our bounded piece in the manifold. Putting the facts together, $\varphi(D) : W^r \to W^{r-m} \hookrightarrow W^r$ is uniformly approximable.

In order to cite [11, theorem 5.5], we need to introduce some notation. Define (global) Sobolev spaces $W^k(S)$ as the completion of $\Gamma_c(S)$ in the norm

$$\|\xi\|_k = (\|s\|^2 + \|Ds\|^2 + \cdots + \|D^k s\|^2)^{1/2}.$$

[2] Recall that $N_\delta(Y)$ denotes the δ–neighborhood of a set Y; and $\Gamma(S)$ denotes the set of smooth sections of S.

Furthermore, if $L \subset M$ we denote

$$||\xi||_{k,L} = \inf\{||\zeta||_k \mid \zeta \in W^k(S), \xi = \zeta \text{ on a nbhd of } L\}.$$

An operator $A : W^k(S) \to W^l(S)$ is called *quasilocal*, if there exists a function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$, such that $\mu(r) \to 0$ as $r \to \infty$ and for each $K \subset M$ and each $\xi \in W^k(S)$ supported within K one has

$$||A\xi||_{l,M\setminus N_r(K)} \leq \mu(r)||\xi||_k.$$

We call μ a dominating function for A. Finally, we set $S^m(\mathbb{R})$ to be the set of functions $\varphi \in C^\infty(\mathbb{R})$, which satisfy inequalities of the form

$$|\varphi^{(k)}(\lambda)| < C_k(1+|\lambda|)^{m-k}$$

and define the Schwartz space $\mathscr{S}(\mathbb{R}) = \bigcap S^m(\mathbb{R})$.

Theorem 3.2 ([11, theorem 5.5]). *Let D be a geometric operator on a Clifford bundle S over a complete manifold M with bounded geometry. If $\varphi \in S^m(\mathbb{R})$, then $\varphi(D)$ continuously extends to a quasilocal operator $W^r(S) \to W^{r-m}(S)$.*

Proof of 3.1. Fix now a function $\varphi \in S^m(\mathbb{R})$ ($m \leq -1$). We are going to show that $\varphi(D)$ is a uniform operator[3]. By the above theorem, there is a dominating function μ for $\varphi(D)$, and $\varphi(D)$ extends to a bounded operator $L^2(S) \to W^{-m}(S)$.

Fix now also $\varepsilon > 0$ and $R > 0$. Pick any open subset $U \subset M$ with $\operatorname{diam}(U) \leq R$. Consider now he restriction of $\varphi(D)$ to sections supported on U (denoted $L^2(S|_U)$). This is sufficient to obtain uniformity, since $\varphi(D)$ is selfadjoint. Since $\mu(r) \to 0$, there is $r_0 > 0$, such that $\mu(r_0) < \varepsilon/2$. Now decompose $\varphi(D)|_{L^2(S|_U)} : L^2(S|_U) \to W^{-m}(S|_{N_{r_0}(U)}) \oplus W^{-m}(S|_{M\setminus N_{r_0}(U)})$. By quasilocality, the second component has norm at most $\varepsilon/2$. It remains to prove that the restrictions of $\varphi(D)$ to $L^2(S|_U) \to W^{-m}(S|_{N_{r_0}(U)}) \hookrightarrow L^2(S|_{N_{r_0}(U)})$ are approximable by finite rank operators, such that the ranks depend only on $\varepsilon > 0$ and $R \geq \operatorname{diam}(U)$.

We can now reduce to the case of a torus with a trivial bundle. This just follows from a partition of unity argument and the existence of nice coordinate patches (from bounded geometry). Also note that for a given R, there is a uniform bound on how many of these patches are needed to cover any subset of M with diameter less than $R + 2r_0$.

On the torus T^n with the trivial bundle $E = T^n \times \mathbb{C}^n$, we can use Fourier series. Denote by $P_N : L^2(E) \to L^2(E)$ the orthogonal projection given by replacing the $\vec{q}$–Fourier coefficient ($\vec{q} \in \mathbb{Z}^n$) of a function by 0 if $|\vec{q}| > N$ (in other words, we truncate the Fourier series at N). The absolute values of the Fourier coefficients of a function in $W^{-m}(E)$ decrease at least as fast as $|\vec{q}|^m$. Consequently, the finite–rank maps $W^{-m}(E) \hookrightarrow L^2(E) \xrightarrow{P_N} L^2(E)$ approximate the inclusion $W^{-m}(E) \to L^2(E)$ in norm for $m \leq -1$. Moreover, for a given $\varepsilon > 0$, the rank of an ε–approximant depends only on ε and m. This concludes the proof of the fact that $\varphi(D)$ is uniform if $\varphi \in S^m(\mathbb{R})$ with $m \leq -1$.

The passage from $\varphi \in S^m(\mathbb{R})$, $m \leq -1$, to $\varphi \in C_0(\mathbb{R})$ is by the usual approximation argument (together with the fact that uniform operators from a C*-algebra, see 4.2). Summarizing, for $\varphi \in C_0(\mathbb{R})$ we have that $\varphi(D)$ is a uniform operator.

Now if χ is any chopping function, then $\chi(D)^2 - 1 = (\chi^2 - 1)(D)$ and $\chi^2 - 1 \in C_0(\mathbb{R})$, hence the Fredholmness condition follows from the previous argument. Furthermore, the difference of two chopping functions is also in $C_0(\mathbb{R})$, and so we are free to choose one particular chopping function (we choose $\chi(t) = \frac{t}{\sqrt{1+t^2}}$) to prove that $\chi(D)$ is l-uniformly pseudolocal. We apply a useful formula from [9, Lemma 4.4]:

$$\chi(D) = \frac{2}{\pi}\int_0^\infty \frac{D}{1+\lambda^2+D^2}\,\mathrm{d}\lambda$$

(convergence in the strong topology), so that

$$[\rho(f), \chi(D)] = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda^2+D^2}\left((1+\lambda^2)[\rho(f), D] + D[\rho(f), D]D\right)\frac{1}{1+\lambda^2+D^2}\,\mathrm{d}\lambda.$$

Fix $\varepsilon > 0$ and $L > 0$. We have estimates

- $||\frac{D}{1+\lambda^2+D^2}|| \leq \frac{1}{2\lambda}$,

[3]Note that the notion of a uniform operator from [11] is different from ours.

- for a smooth $f \in C_{R,L}(M)$, $[\rho(f), D]$ is the multiplication operator by the derivative of f, and so we have that $||[\rho(f), D]|| \leq L$.

Consequently, the integral in the last display converges in norm; and there exists $k > 0$ (depending only on R and L) and $\lambda_1, \dots, \lambda_k$, such that the integral can be approximated within $\varepsilon > 0$ by the sum of the integrands with $\lambda = \lambda_1, \dots, \lambda_k$. Now each of the operators $\frac{D}{1+\lambda^2+D^2}$, $\frac{1}{1+\lambda^2+D^2}$ is uniform by the previous considerations: $\frac{t}{1+\lambda^2+t^2}$, $\frac{1}{1+\lambda^2+t^2} \in S^{-1}(\mathbb{R})$. This finishes the proof. □

We finish the section by an observation, which can be applied to obtain uniform Fredholm modules for non–geometric elliptic operators. We assume that a finitely generated discrete group Γ acts cocompactly on M (this assumption implies that M has bounded geometry), and that D commutes with this action. The vague reason for uniformity is that D "looks the same" on each translate of a fundamental domain (which is bounded), and so the approximation properties of D at any place of M are the same as those over a fixed fundamental domain. In this case, just knowing that $\varphi(D)$ is locally compact for $\varphi \in C_0(R)$ upgrades to:

Claim 1. For any $\varphi \in C_0(\mathbb{R})$, the operator $\varphi(D)$ is uniform.

Proof. For a given $R > 0$, we can find a bounded open set $U \subset M$, such that the collection $\{U\gamma\}_{\gamma\in\Gamma}$ covers M and has Lebesgue number at least R. Construct a continuous function $f : M \to [0, 1]$, which is 1 on $\overline{U}$ and 0 outside a small neighborhood of U. Then for any function $g \in C_R(M)$ there is a $\gamma \in \Gamma$, such that $g \cdot f^\gamma = g$ (by f^γ we denote the translate of f by γ). Then $\rho(g)\varphi(D) = \rho(gf^\gamma)\varphi(D) = \rho(g)\rho(f^\gamma)\varphi(D^\gamma) = \rho(g)(\rho(f)\varphi(D))^\gamma$. Hence (ε, N)–approximability of $\rho(g)\varphi(D)$ is not worse than the one of $\rho(f)\varphi(D)$ (which is a compact operator, independent of g). This proves that $\varphi(D)$ is uniform. □

Pseudolocality can be now deduced in the same way as in the geometric case from the claim, provided that $||[\rho(f), D]||$ is bounded independently of $f \in C_{R,L}(M)$.

4. Dual algebras

In the analytic K-homology, one can use the Voiculescu's theorem and a standard normalizing procedure to express K-homology as a K-theory of a certain C*-algebra. In this section, we first work on a fixed X-module (H, ϕ) to obtain a similar isomorphism for the "partial" uniform K-homology groups (proposition 4.3). To work around the Voiculescu's theorem, we express the uniform K-homology as a direct limit of "partial" uniform K-homology groups (proposition 4.9).

Definition 4.1 (Dual algebras)**.** Let H be a Hilbert space and $\phi : C_0(X) \to \mathscr{B}(H)$ be a *-representation. We define $\Psi^0_\phi(X) \subset \mathscr{B}(H)$ to be the set of all l-uniformly pseudolocal operators in $\mathscr{B}(H)$ and $\Psi^{-1}_\phi(X) \subset \mathscr{B}(H)$ to be the set of all uniform operators. Furthermore, we denote $\mathscr{D}^u_\phi(X) = \Psi^0_{\phi\oplus 0}(X) \subset \mathscr{B}(H \oplus H)$.

Lemma 4.2. *Let H be a Hilbert space and $\phi : C_0(X) \to \mathscr{B}(H)$ a *-representation. Then $\Psi^0_\phi(X) \subset \mathscr{B}(H)$ is a C*-algebra. Likewise, $\Psi^{-1}_\phi(X) \subset \Psi^0_\phi(X)$ is a C*-algebra. Furthermore, $\Psi^{-1}_\phi(X)$ is a closed two-sided ideal of $\Psi^0_\phi(X)$.*

Proof. We show that $\Psi^0_\phi(X)$ is norm-closed. Assume that $T \in \mathscr{B}(H)$ is approximable by l-uniformly pseudolocal operators. Take $\varepsilon > 0$ and $R, L \geq 0$. By assumption, there is an l-uniformly pseudolocal operator $S \in \mathscr{B}(H)$, such that $||T - S|| < \varepsilon/4$. Let M be such that S is $(\varepsilon/2, R, L, M; \phi)$–approximable. Hence for any $f \in C_{R,L}(X)$ there exists $k \in \mathscr{B}(H)$ with $\mathrm{rank}(k) \leq M$ such that $||[\phi(f), S] - k|| < \varepsilon/2$. Consequently, $||[\phi(f), T] - k|| \leq ||[\phi(f), (T - S)]|| + ||[\phi(f), S] - k|| < \varepsilon$. In other words, $[\phi(f), T]$ is (ε, M)–approximable. The proof that the norm-limits of uniform operators are again uniform is analogous.

The fact that $\Psi^0_\phi(X)$ is closed under multiplication follows from the identity $[\phi(f), ST] = [\phi(f), S]T + S[\phi(f), T]$. Likewise, using the identity $\phi(f)ST = [\phi(f), S]T + S\phi(f)T$ we obtain that $\Psi^{-1}_\phi(X)$ is an ideal of $\Psi^0_\phi(X)$ (we're using remark 2.5 here). □

For a fixed X-module (H, ϕ), define a group $K^u_*(X;\phi)$ in a similar manner as $K^u_*(X)$, except that we consider only (unitary equivalence classes of) uniform Fredholm modules, whose Hilbert spaces and $C_0(X)$-actions are direct sums (finite or countably infinite) of $(H \oplus H, \phi \oplus 0)$. A glance at the proofs for $K^u_*(X)$ shows that $K^u_*(X;\phi)$ can be characterized also as a group of (unitary equivalence classes of) uniform Fredholm modules over the sums of $(H \oplus H, \phi \oplus 0)$, with homotopies also taken within this category (see 2.14).

Fix (H,ϕ) for a time being, and let us define a homomorphism

$$\varphi_0 : K_1(\mathscr{D}^u_\phi(X)) \to K^u_0(X;\phi)$$

as follows: If $U \in \mathscr{M}_n(\mathscr{D}^u_\phi(X))$ is a unitary representing a K_1-class, we set $\varphi_0([U]) = [(H^{2n},(\phi\oplus 0)^n,U)]$. It is immediate that $(H^{2n},(\phi\oplus 0)^n,U)$ is a uniform Fredholm module. Since homotopies of unitaries translate into operator homotopies of Fredholm modules and the operations on K_1 and K^u_0 are both direct sums, we see that φ_0 is a group homomorphism.

Analogously, we induce a homomorphism

$$\varphi_1 : K_0(\mathscr{D}^u_\phi(X)) \to K^u_1(X;\phi)$$

by assigning to a projection $Q \in \mathscr{M}_n(\mathscr{D}^u_\phi(X))$ the triple $(H^{2n},(\phi\oplus 0)^n, 2Q-1)$. It is again easy to check that this triple is actually a uniform 1-Fredholm module. Since operations on K_0 and K^u_1 are both direct sums and homotopies translate to homotopies, we really do get a group homomorphism.

Proposition 4.3 ("One X-module" picture)**.** *The above defined maps $\varphi_* : K_{1-*}(\mathscr{D}^u_\phi(X)) \to K^u_*(X;\phi)$ are isomorphisms.*

The proof follows the usual route of showing that elements of $K^u_*(X;\phi)$ have nice representatives (cf. [8, sections 8.3 and 8.4]). It is done by the following three lemmas.

Lemma 4.4. *Any element of $K^u_*(X;\phi)$ may be represented by a uniform Fredholm module of the form $(H^{2n},(\phi\oplus 0)^n,S)$, where $||S||\le 1$. Furthermore, homotopies can be also assumed to have this property.*

Proof. This is a standard cutting argument. We first deal with the even case. Take any representative $(H^{2n},(\phi\oplus 0)^n,S)$. Consider the matrix $\tilde{S} = \left(\begin{smallmatrix} 0 & S \\ S^* & 0\end{smallmatrix}\right)$. It represents an odd selfadjoint operator in $\mathscr{B}(H^{4n})$, whose square differs from 1 by a uniform operator. Take the cutting function $c:\mathbb{R}\to\mathbb{R}$ given by

$$c(t) = \begin{cases} -1 & \text{if } t < -1 \\ t & \text{if } -1 \le t \le 1 \\ 1 & \text{if } t > 1, \end{cases}$$

By functional calculus, $c(\tilde{S})$ is again an odd selfadjoint operator (since c is odd), but with $||c(\tilde{S})|| \le 1$. Denote by T the upper right corner of $c(\tilde{S})$. Then $||T||\le 1$, and $T-S$ is uniform. The last statement can be seen by referring to the theorem on the essential spectrum of selfadjoint operators. The proof is completed by applying Lemma 2.16.

The odd case is even more straightforward, since we may take a representative $(H^{2n},(\phi\oplus 0)^n,P)$ with $P=P^*$. Hence we can apply the cutting directly to P and replace it by $c(P)$.

The same procedures can be applied to whole homotopies. □

Lemma 4.5. *Any element of $K^u_0(X;\phi)$ may be represented by a uniform 0-Fredholm module of the form $(H^{2n},(\phi\oplus 0)^n,S)$, where S is a unitary. Furthermore, the homotopies can also be assumed to have this property.*

Proof. Take a representative $(H^{2n},(\phi\oplus 0)^n,S)$, such that $||S||\le 1$. For simplicity, assume $n=1$, so that $S = \left(\begin{smallmatrix} T & S_{12} \\ S_{21} & S_{22}\end{smallmatrix}\right)$, $T, S_{ij}\in\mathscr{B}(H)$. It follows that $||T||\le 1$, so the operator $U = \left(\begin{smallmatrix} T & -\sqrt{1-TT^*} \\ \sqrt{1-T^*T} & T^*\end{smallmatrix}\right)$ is well defined and unitary.

Since S is l-uniformly pseudolocal, T is l-uniformly pseudolocal and for any $\varepsilon>0$, $R,L\ge 0$ there exists $M>0$, such that $\phi(f)S_{12}$ and $S_{21}\phi(f)$ are (ε,M)–approximable for all $f\in C_{R,L}(X)$. Using this and uniformity of $1-SS^*$ and $1-S^*S$, we conclude that $1-T^*T$ and $1-TT^*$ are uniform. Since $\Psi^{-1}_\phi(X)$ is a C*-algebra, so are their square roots. Consequently, $S-U$ is uniform, and another application of Lemma 2.16 finishes the proof.

Again, we can apply this procedure to the whole homotopy. □

Lemma 4.6. *Any class in $K^u_1(X;\phi)$ can be represented by a uniform 1-Fredholm module of the form $(H^{2n},(\phi\oplus 0)^n,P)$, where $P^2=1$.*

Proof. We proceed similarly as in the previous lemma. Take a representative $(H^{2n}, (\phi \oplus 0)^n, P)$, such that $P = P^*$ and $\|P\| \le 1$. For simplicity, we assume that $n = 1$, and so $P = \begin{pmatrix} Q & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$, where $Q, P_{ij} \in \mathcal{B}(H)$. It follows that Q is also selfadjoint and contractive. Therefore, the operator $O = \begin{pmatrix} Q & \sqrt{1-Q^2} \\ \sqrt{1-Q^2} & -Q \end{pmatrix}$ is selfadjoint with $O^2 = 1$.

As in the previous proof, we obtain that $1 - Q^2$ is uniform and that $P - O$ is uniform as well. This finishes the proof. □

Let us now turn to relationship between $K_*^u(X, \phi)$'s for different ϕ's. We shall need another definition (which is more general than what we need at the moment, but full generality will be required later):

Definition 4.7. Let X and Z be spaces, let $\varphi : C_0(X) \to C_0(Z)$ be a *-homomorphism, $\phi_X : C_0(X) \to \mathcal{B}(H_X)$ and $\phi_Z : C_0(Z) \to \mathcal{B}(H_Z)$ be *-representations. We say that an isometry $V : H_Z \to H_X$ *uniformly covers* φ, if for every $\varepsilon > 0$, $R, L \ge 0$ there exists $M \ge 0$, such that $V^* \phi_X(f) V - \phi_Z(\varphi(f))$ is (ε, M)–approximable for every $f \in C_{R,L}(X)$. In short, $V^* \phi_X(\cdot) V \sim_{lua} \phi_Z(\varphi(\cdot))$.

We introduce a relation $\prec$ on the set $\mathcal{X}$ of (unitary equivalence classes of) *-representations ϕ of $C_0(X)$ on some (separable) Hilbert space, which turns it into a directed system. We define the relation $\prec$ by declaring that $(H, \phi) \prec (E, \rho)$ (or just $\phi \prec \rho$) if and only if there exists an isometry $V_{\phi,\rho} : H \to E$ which uniformly covers the identity map $\mathrm{id} : C_0(X) \to C_0(X)$. The reflexivity of $\prec$ is obvious and the transitivity becomes clear after a momentary reflection on the definition of uniform covering. Furthermore, for $\phi, \rho \in \mathcal{X}$, we easily see that $\phi \prec \phi \oplus \rho$ and $\rho \prec \phi \oplus \rho$.

If $\phi \prec \rho$, then we obtain a homomorphism

$$i_{V_{\phi,\rho}} : K_*^u(X, \phi) \to K_*^u(X, \rho)$$

using Proposition 4.3 and the fact that $\mathrm{Ad}(V_{\phi,\rho})$ maps $\Psi_\phi^0(X)$ into $\Psi_\rho^0(X)$ (where $\mathrm{Ad}(V)$ is defined as $\mathrm{Ad}(V)(T) = VTV^*$ and it's a *-homomorphism when V is an isometry). This fact is a special case (when $Z = X$ and $\pi = \mathrm{id}$) of Lemma 5.4 from the next section, where we prove a more general statement requiring new notation.

The set of $K_*^u(X, \phi)$'s, together with the maps $i_{V_{\phi,\rho}}$, becomes a directed system indexed by $\mathcal{X}$. The next lemma ensures that we may arbitrarily choose (and fix that choice of) an isometry $V_{\phi,\rho}$ for each pair $\phi \prec \rho$.

Lemma 4.8. *We adopt the notation from Definition 4.7. If two isometries $V_1, V_2 : H_Z \to H_X$ uniformly cover φ, then the induced maps on K-theory are the same:*

$$(\mathrm{Ad}(V_1))_* = (\mathrm{Ad}(V_2))_* : K_*(\Psi_{\phi_Z}^0(Z)) \to K_*(\Psi_{\phi_X}^0(X)).$$

(Note that by the proof of Lemma 5.4, $\mathrm{Ad}(V_i)$*'s really map* $\Psi_{\phi_Z}^0(Z)$ *into* $\Psi_{\phi_X}^0(X)$*.)*

This lemma is analogous to the second part of [8, Lemma 5.2.4] and the proof carries over verbatim. This lemma also implies that $\prec$ becomes antisymmetric when it descends to $K_*^u(X, \phi)$'s.

For each ϕ there is an obvious homomorphism $j_\phi : K_*^u(X; \phi) \to K_*^u(X)$. It is also clear that j_ϕ's commute with $i_{V_{\phi,\rho}}$'s, which allows us to state the final proposition of this section:

Proposition 4.9 (Direct limit version)**.** *With the notation above,*

$$K_*^u(X) = \lim_{\phi \in \mathcal{X}} j_\phi(K_*^u(X, \phi)).$$

5. Mayer–Vietoris sequence

The goal of this section is to prove the Mayer–Vietoris sequence for uniform K-homology groups:

Theorem 5.1 (Mayer–Vietoris sequence)**.** *Let $A, B \subset X$ be closed subsets of X, such that $A \cup B = X$, $\mathrm{int}(A \cap B) \neq \emptyset$ and $d(A \setminus B, B \setminus A) > 0$.*[4] *Then there is a 6-term exact sequence*

$$\begin{array}{ccccc} K_0^u(A\cap B) & \longrightarrow & K_0^u(A)\oplus K_0^u(B) & \longrightarrow & K_0^u(X) \\ \uparrow & & & & \downarrow \\ K_1^u(X) & \longleftarrow & K_1^u(A)\oplus K_1^u(B) & \longleftarrow & K_1^u(X). \end{array}$$

Before outlining the proof, we need a definition:

Definition 5.2. Given a Hilbert space H and a *-representation $\phi : C_0(X) \to \mathcal{B}(H)$, we let $\Psi^0_\phi(X, Z) \subset \Psi^0_\phi(X)$ to be the set of all operators $T \in \Psi^0_\phi(X)$ which are uniform on $X \setminus Z$, that is, such that for every $\varepsilon > 0, R \geq 0$, there exists $M > 0$, such that for every $f \in C_R(X)$ with $f|_Z = 0$ we have that $\phi(f)T$ and $T\phi(f)$ are (ε, M)–approximable. Also, we set $\mathcal{D}^u_\phi(X, Z) = \Psi^0_{\phi\oplus 0}(X, Z) \subset \mathcal{B}(H \oplus H)$.

Note that a proof similar to the proof of Lemma 4.2 yields that $\Psi^0_\phi(X, Z)$ is a closed two-sided ideal of $\Psi^0_\phi(X)$.

Proof of 5.1. The strategy is to first use the C*-algebra Mayer–Vietoris sequence (with ϕ fixed), and then apply Propositions 4.3, 4.9 and Excision Lemma 5.3 to obtain the result.

Keeping the notation from 5.1, we have that $\mathcal{D}^u_\phi(X, A) \cap \mathcal{D}^u_\phi(X, B) = \mathcal{D}^u_\phi(X, A \cap B)$ directly from the definitions, and $\mathcal{D}^u_\phi(X, A) + \mathcal{D}^u_\phi(X, B) = \mathcal{D}^u_\phi(X)$ (by a partition of unity argument[5]). Subsequently, from the C*-algebra Mayer–Vietoris sequence, we get that

$$\begin{array}{ccccc} K_0(\mathcal{D}^u_\phi(X, A\cap B)) & \longrightarrow & K_0(\mathcal{D}^u_\phi(X,A))\oplus K_0(\mathcal{D}^u_\phi(X,B)) & \longrightarrow & K_0(\mathcal{D}^u_\phi(X)) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{D}^u_\phi(X)) & \longleftarrow & K_1(\mathcal{D}^u_\phi(X,A))\oplus K_1(\mathcal{D}^u_\phi(X,B)) & \longleftarrow & K_1(\mathcal{D}^u_\phi(X, A\cap B)) \end{array}$$

is exact.

The general Mayer–Vietoris sequence now follows by "taking the direct limit", i.e. using naturality of our constructions, Proposition 4.9 and Excision Lemma 5.3. □

It remains to deal with the excision lemma. For the rest of this section, we shall denote by X a proper metric space, by $Z \subseteq X$ a closed subset of X and by $\pi : C_0(X) \to C_0(Z)$ the restriction homomorphism.

Lemma 5.3 (Excision lemma)**.** *There is a natural isomorphism*

$$\lim_{\phi} K_*(\mathcal{D}^u_\phi(X, Z)) \cong \lim_{\phi_Z} K_*(\mathcal{D}^u_{\phi_Z}(Z)).$$

By virtue of 4.9, we may say that the "relative uniform K-homology" $K^u_*(X, Z)$ is isomorphic to $K^u_*(Z)$.

Proof. The strategy is obtain a commutative diagram (notation will be introduced in the course of the proof)

$$\begin{array}{ccccccc} & & K_*(\Psi^0_{\phi_X}(X,Z)) & \xrightarrow{Ad(SW)} & K_*(\Psi^0_{\phi'_X}(X,Z)) & \longrightarrow & \\ & \overset{\mathrm{Ad}(V)}{\nearrow} & & \overset{\mathrm{Ad}(W)}{\searrow} \quad \overset{\mathrm{Ad}(S)}{\nearrow} & & \searrow & \\ K_*(\Psi^0_{\phi_Z}(Z)) & & \xrightarrow{\mathrm{Ad}(WV)} & K_*(\Psi^0_{\phi'_Z}(Z)) & & \longrightarrow & \end{array} \tag{1}$$

starting with the following data: a representation $\phi_X : C_0(X) \to \mathcal{B}(H_X)$, a representation $\phi_Z : C_0(Z) \to \mathcal{B}(H_Z)$ and an isometry $V : H_Z \to H_X$, which uniformly covers π (this gives the first $\nearrow$ in (1)). In the diagram, the horizontal arrows shall uniformly cover the identity (on the level of K-theory), and the diagonals heading up will uniformly cover π. This would establish the lemma.

[4] This last condition just expresses the requirement that "the overlap of A and B does not get arbitrarily thin". It is used only in the next footnote.

[5] Take $f, g \in C_b(X)$ with $f + g = 1$, $f|_{X\setminus A} = 0$ and $g|_{X\setminus B} = 0$, f, g are L-continuous for some L (this is possible since $d(A \setminus B, B \setminus A) > 0$). Write $T = T\phi(f) + T\phi(g)$. Now if $h|_A = 0$, then $T\phi(f)\phi(h) = 0$ and $\phi(h)T\phi(f) = [\phi(h), T]\phi(f) + T\phi(h)\phi(f)$.

Let us now explain how can we arrange the starting data. If we start with a $*$-representation $\phi_X : C_0(X) \to \mathscr{B}(H_X)$, it induces a Borel measure on X, and extends to a $*$-representation (also denoted by ϕ_X) of $\ell^\infty(X)$. In particular, we may restrict ϕ_X to a representation $\phi_Z : C_0(Z) \to \mathscr{B}(H_X)$ and let $H_Z = \chi_Z H_X$. Then the inclusion $V : H_Z \hookrightarrow H_X$ actually exactly covers π, i.e. $V^*\phi_X(f)V = \phi_Z(\pi(f)) = \phi(\chi_Z f)$ for all $f \in C_0(X)$.

Conversely, starting with a $*$-representation $\phi_Z : C_0(Z) \to \mathscr{B}(H_Z)$, we obtain a $*$-representation $\phi_X = \phi_Z \circ \pi$ of $C_0(X)$, so that we can put $H_X = H_Z$ and $V = \mathrm{id}$.

The rest of the proof is devoted to obtaining a diagram (1) from given ϕ_X, ϕ_Z and V uniformly covering π. We accomplish our goal similarly as [8, proof of 3.5.7].

Lemma 5.4. *Let $\phi_X : C_0(X) \to \mathscr{B}(H_X)$, $\phi_Z : C_0(Z) \to \mathscr{B}(H_Z)$ be *-representations and let $V : H_Z \to H_X$ be an isometry which uniformly covers π. Then*

$$\mathrm{Ad}(V)(\Psi^0_{\phi_Z}(Z)) \subset \Psi^0_{\phi_X}(X, Z).$$

(The adjoint map Ad *is defined as* $\mathrm{Ad}(V)(T) = VTV^*$*, and it's a *-homomorphism since V is an isometry.)*

Proof. We first show that $VV^* \in \Psi^0_{\phi_X}(X, Z)$. Decompose $H_X = VV^*H_X \oplus (1 - VV^*)H_X$. With respect to this decomposition $VV^* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and we denote $\phi_X = \left(\begin{smallmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{smallmatrix}\right)$. The fact that VV^* is ϕ_X–uniformly pseudolocal is equivalent to

$$\phi_{12}(\cdot) \text{ and } \phi_{21}(\cdot) \text{ are l-uniformly approximable.}$$

Using the covering assumption,

$$\begin{aligned} \phi_{11}(f^*f) = VV^*\phi_X(f^*f)VV^* \sim_{lua} V\phi_Z(\pi(f^*f))V^* = V\phi_Z(\pi(f))^*\phi_Z(\pi(f))V^* \sim_{lua} \\ \sim_{lua} VV^*\phi_X(f)^*VV^*\phi_X(f)VV^* = \phi_{11}(f)^*\phi_{11}(f). \end{aligned}$$

Since ϕ_X is a *-homomorphism, we have

$$\phi_{21}(f)^*\phi_{21}(f) = \phi_{11}(f^*f) - \phi_{11}(f)^*\phi_{11}(f) \tag{2}$$

for each $f \in C_0(X)$. In other words, $\phi_{21}(\cdot)^*\phi_{21}(\cdot)$ is l-uniformly approximable. Using the spectral theorem for compact selfadjoint operators[6], also $\sqrt{\phi_{21}(\cdot)^*\phi_{21}(\cdot)} = |\phi_{21}(\cdot)|$ is l-uniformly approximable. Let $\phi_{21}(f) = u(f)|\phi_{21}(f)|$ denote the polar decomposition. From this formula, it follows that $\phi_{21}(f)$ is l-uniformly approximable as well.

To show that VV^* is uniform on $X \setminus Z$, it suffices to observe that in addition to $\phi_{12}(\cdot)$ and $\phi_{21}(\cdot)$ being l-uniformly approximable, we also have $\phi_{11}(f) = VV^*\phi_X(f)VV^* \sim_{lua} V\phi_Z(\pi(f))V^* = 0$ for $f \in C_0(X \setminus Z)$.

We have shown that $VV^* \in \Psi^0_{\phi_X}(X, Z)$. From this, we easily get that $\mathrm{Ad}(V)$ maps $\Psi^0_{\phi_Z}(Z)$ into $\Psi^*_{\phi_X}(X, Z)$. ☐

Let $\sigma : C_0(Z) \to C_0(X)$ be a completely positive lift of π that satisfies

- if $f \in C_R(X)$ then $\mathrm{supp}(\sigma(\pi(f))) \subset \{x \in X \mid d(x, \mathrm{supp}(f)) \le 1\}$,
- there exists L', such that if f is L-continuous then $\sigma(f)$ is $L + L'$-continuous.

In particular, if $g \in C_R(Z)$ then $\sigma(g) \in C_{R+2}(X)$. Such a lift exists.[7] Now $\phi_X\sigma : C_0(Z) \to \mathscr{B}(H_X)$ is a completely positive map, so by the Stinespring's theorem, there exist a Hilbert space H and maps $\rho_{12}, \rho_{21}, \rho_{22}$ such that

$$\phi'_Z = \begin{pmatrix} \phi_X\sigma & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} : C_0(Z) \to \mathscr{B}(H_X \oplus H)$$

is a *-homomorphism. Denote by $W : H_X \to H_X \oplus H$ the obvious inclusion.

Claim 2. $\mathrm{Ad}(W)$ maps $\Psi^0_{\phi_X}(X, Z)$ into $\Psi^0_{\phi'_Z}(Z)$. Furthermore WV uniformly covers $\mathrm{id} : C_0(Z) \to C_0(Z)$. In other words, $W^*V^*\phi'_Z(\cdot)VW - \phi_Z(\cdot)$ is l-uniformly approximable on $C_0(Z)$.

---

[6] If $k \in \mathscr{K}$ is selfadjoint, then for $\varepsilon > 0$ we can approximate k by a rank-M operator, where M is the sum of dimensions of eigenspaces corresponding to all eigenvalues λ with $|\lambda| > \varepsilon$.

[7] Note that a positive map between commutative C*-algebras is automatically completely positive, and a nice positive linear lift can be constructed using a linear basis and the Urysohn lemma–type construction. The L-continuity can be also arranged.

Proof. Decomposing into matrices shows that $\operatorname{Ad}(W)(T)$ belongs to $\Psi^0_{\phi'_Z}(Z)$ if and only if $T\rho_{12}(\cdot)$ and $\rho_{21}(\cdot)T$ are l-uniformly approximable. Since ϕ'_Z is a *-homomorphism, $\rho_{21}(f)^*\rho_{21}(f) \in \phi_X(C_0(X \setminus Z))$ for all $f \in C_0(Z)$, cf. (2). Hence $\rho^*_{21}(f)\rho_{21}(f)T$ is l-uniformly approximable. Consequently, $T^*\rho^*_{21}(f)\rho_{21}(f)T = (\rho_{21}(f)T)^*(\rho_{21}(f)T)$ is l-uniformly approximable as well, and it follows by the argument in the proof of Lemma 5.4 that $\rho_{21}(f)T$ itself is as well. This finishes the first part.

To see that WV uniformly covers id on $C_0(Z)$, just observe that for $f \in C_0(Z)$, we have $V^*W^*\phi'_Z(f)WV = V^*\phi_X(\sigma(f))V \sim_{lua} \phi_Z(\pi(\sigma(f))) = \phi_Z(f)$ by assumption of V. □

The next step is to consider the Hilbert space $H'_X = H_X \oplus (H_X \oplus H)$ with the *-representation $\phi'_X = \phi_X \oplus \phi'_Z\pi$ of $C_0(X)$. Denote by $S : H_X \oplus H \to H'_X$ the inclusion (H_X is included as the second H_X summand).

Claim 3. S uniformly covers π. $\operatorname{Ad}(SW)$ is homotopic to a *-homomorphism which uniformly covers $\mathrm{id}: C_0(X) \to C_0(X)$. Hence we are in the position to iterate the construction we have done so far to obtain a commutative diagram (1).

Proof. In fact, S actually exactly covers π, since $S^*\phi'_X S = \phi'_Z\pi$. Continuing with the second part of the claim, note that SW includes H_X into $H_X \oplus H_X \oplus H$ as the second copy of H_X. If we denote by $Y : H_X \to H_X \oplus H_X \oplus H$ the inclusion as the first summand, then $\operatorname{Ad}(Y)$ exactly covers $\mathrm{id} : C_0(X) \to C_0(X)$. Furthermore, $\operatorname{Ad}(SW)$ and $\operatorname{Ad}(Y)$ are homotopic via the homotopy of *-homomorphisms

$$A_t : T \mapsto \begin{pmatrix} \sin^2(\frac{\pi}{2}t)T & \sin(\frac{\pi}{2}t)\cos(\frac{\pi}{2}t)T & 0 \\ \sin(\frac{\pi}{2}t)\cos(\frac{\pi}{2}t)T & \cos^2(\frac{\pi}{2}t)T & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad t \in [0,1].$$

It remains to verify that A_t maps $\Psi^0_{\phi_X}(X,Z)$ into $\Psi^0_{\phi'_X}(X,Z)$. To this end, it is enough to observe that if $T \in \Psi^0_{\phi_X}(X,Z)$, then $\tilde T = \left(\begin{smallmatrix} T & T & 0 \\ T & T & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in \mathcal{B}(H_X \oplus H_X \oplus H)$ is ϕ'_X–l-uniformly pseudolocal and uniform on $C_0(X \setminus Z)$. For $f \in C_0(X)$, we compute

$$[\tilde T, \phi'_X(f)] = \begin{pmatrix} T\phi_X(f) - \phi_X(f)T & T\phi_X\sigma\pi(f) - \phi_X(f)T & 0 \\ T\phi_X(f) - \phi_X\sigma\pi(f)T & T\phi_X\sigma\pi(f) - \phi_X\sigma\pi(f)T & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is now clear that for showing l-pseudolocality of $\tilde T$ it suffices to see that $T\phi_X(f) - \phi_X\sigma\pi(f)T = [T, \phi_X(f)] + (\phi_X(f - \sigma\pi(f)))T$ is l-uniformly approximable. But $f - \sigma\pi(f) \in C_0(X \setminus Z)$, hence the assertion follows from the assumptions on T and the lift σ.

Similarly $\tilde T\phi'_X(f) = \left(\begin{smallmatrix} T\phi_X(f) & T\phi_X\sigma\pi(f) & 0 \\ T\phi_X(f) & T\phi_X\sigma\pi(f) & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and the uniformness of $\tilde T$ on $C_0(X \setminus Z)$ follows from the observation that $\pi(f) = 0$ for $f \in C_0(X \setminus Z)$. □

This finishes the proof of Lemma 5.3. □

6. Coarse geometry and C*-algebras

The first part of this section is devoted to a review of basic notions from coarse geometry. The second part recalls the definitions of C*-algebras reflecting the coarse structure: (uniform) Roe C*-algebras.

Coarse geometry studies large–scale behavior of spaces. While it is possible to give an abstract definition of a coarse structure (see [13]), for our purposes it is sufficient and more straightforward to assume that our spaces are endowed with a metric. The appropriate notion of maps in the "coarse category" is the following:

Definition 6.1 (Coarse maps)**.** A (not necessarily continuous) map $g : X \to Z$ between metric spaces X and Z is said to be *coarse*, if:

- For any $r \geq 0$ there exists $R \geq 0$, such that $d(x_1, x_2) \leq r$ implies $d(g(x_1), g(x_2)) \leq R$ for $x_{1,2} \in X$. An equivalent condition is that there exists a non-decreasing function $\rho^+ : \mathbb{R}^+ \to \mathbb{R}^+$, such that $d(g(x_1), g(x_2)) \leq \rho^+(d(x_1, x_2))$.
- For any $r \geq 0$ we have $\operatorname{diam}(g^{-1}(B(z,r))) < \infty$ for all $z \in Z$. This condition is referred to as being *cobounded*.

Furthermore, we say that g is called *uniformly cobounded*, if for any $r \geq 0$, we have

$$R_g(r) := \sup_{z \in Z} \operatorname{diam}(g^{-1}(B(z,r))) < \infty.$$

When working in the "coarse category", we may choose a nice representative in the class of coarsely equivalent spaces:

Definition 6.2. A metric space Y is said to be *uniformly discrete*, if there is $\delta > 0$, such that $d(x,y) \geq \delta$ whenever $x \neq y \in Y$.

Furthermore, Y is said to have *bounded geometry*, if for any $r \geq 0$ we have

$$\sup_{y\in Y} |B(y,r)| < \infty.$$

When switching between discrete and "continuous spaces", the following concept proved to be useful:

Definition 6.3 (Rips complex)**.** Let X be a metric space and let $d \geq 0$. The *Rips complex* $P_d(X)$ is a simplicial polyhedron defined as follows:

- the vertex set of $P_d(X)$ is X,
- any $q+1$ vertices $x_0, x_1, \dots, x_q$ span a simplex of $P_d(X)$ if and only if

$$d(x_i, x_j) \leq d, \qquad \forall i,j \in \{0,\dots,q\}.$$

Note that if X has bounded geometry, $P_d(X)$ is locally finite and finite dimensional. We endow it with the geodesic metric.

We now define C*-algebras, which reflect large–scale behavior of metric spaces. Let Y be a uniformly discrete metric space with bounded geometry. We consider the Hilbert space $\ell^2(Y) \otimes \ell^2(\mathbb{N}) \cong \ell^2(Y \times \mathbb{N})$ (or $\ell^2(Y)$), and represent bounded operators T on it as matrices $T = (t_{yx})_{x,y\in Y}$ with entries t_{yx} in $\mathcal{B}(\ell^2(\mathbb{N}))$ (or $\mathbb{C}$ respectively).

Definition 6.4 (Finite propagation: discrete version)**.** We say that $T = (t_{yx}) \in \mathcal{B}(\ell^2(Y \times \mathbb{N}))$ (or $\mathcal{B}(\ell^2(Y))$) has *finite propagation*, if there exists $R \geq 0$, such that $t_{yx} = 0$ whenever $d(x,y) > R$. The smallest such R is called the *propagation* of T and denoted by propagation(T).

Definition 6.5. We say that T is locally compact, if $t_{yx} \in \mathcal{K}(\ell^2(\mathbb{N}))$ for all $x,y \in Y$. (This condition is void in the case $T \in \mathcal{B}(\ell^2(Y))$.)

We say that T has *uniformly bounded coefficients*, if there exists $C > 0$, such that $||t_{yx}|| \leq C$ for all $x,y \in Y$.

Definition 6.6. The norm-closure of the algebra of all finite propagation operators with uniformly bounded coefficients in $\mathcal{B}(\ell^2(Y))$ is said to be the *uniform Roe C*-algebra of* Y, denoted by C_u^*Y.

We denote by $C_k^*(Y)$ the norm-closure of the algebra of all locally compact finite propagation operators $T = (t_{yx})$ with uniformly bounded coefficients in $\mathcal{B}(\ell^2(Y \times \mathbb{N}))$, which satisfy the additional condition that the set $\{t_{yx} \mid x,y \in Y\} \subset \mathcal{K}(\ell^2(\mathbb{N}))$ is compact in the norm topology on $\mathcal{K}(\ell^2(\mathbb{N}))$.

Remark 6.7. The additional condition in the previous definition merely says that up to ε, we have only finitely many entries t_{yx}.

Another way of stating this condition is that for each $\varepsilon > 0$ there exists $M \geq 0$, such that each t_{xy}, $x,y \in Y$, is at distance at most ε from a rank-M operator.

Remark 6.8. The C*-algebra C_u^*Y is not functorial under coarse uniformly cobounded maps, as an examples of one-point and two-point spaces show. Nevertheless, coarsely equivalent spaces have Morita equivalent uniform Roe C*-algebras, see [5]. This corresponds to the fact that $C_k^*(Y)$ is functorial under such maps.

We now cite a proposition, which provides an estimate on the norm of an operator in terms of its entries:

Proposition 6.9 (see [13])**.** *Let Y be a uniformly discrete space with bounded geometry, and let $t = (t_{yz})_{y,z\in Y}$ be a matrix with entries $t_{yz} \in \mathcal{K}(H)$ [or $t_{yz} \in \mathbb{C}$]. For every $P > 0$ there is $C > 0$, such that if t has propagation at most P, we have $||t|| \leq C \sup_{y,z} ||t_{yz}||$, with the operator norm in $\mathcal{B}(\ell^2 Y \otimes H)$ [or $\mathcal{B}(\ell^2 Y)$ respectively].*

To finish the section, we show that as far as K-theory of uniform Roe algebras is concerned, we may work with $C_k^*(Y)$.

Lemma 6.10. *Let Y be a uniformly discrete metric space with bounded geometry. Then $C_k^*(Y) \cong C_u^*Y \otimes \mathcal{K}$.*

Proof. We show that $C_u^*Y \otimes \mathcal{K}(\ell^2(\mathbb{N}))$ is dense in $C_k^*(Y)$ (with the obvious inclusion). Pick $T = (t_{yx}) \in C_k^*(Y)$ and $\epsilon > 0$. Denote the propagation of T by p. By proposition 6.9, there is a constant $C > 0$, such that if $S = (s_{yx})$ is a matrix of compacts with propagation at most p, then $\|S\| \leq C \sup_{x,y\in X} \|s_{yx}\|$. Since $\{t_{yx} \mid x,y \in Y\}$ is compact, there is an ϵ/C-net $t_1,\dots,t_m$ in it. Then clearly T is ϵ-far from an operator of the form $T_1 \otimes t_1 + \cdots + T_m \otimes t_m$, where each $T_i \in C_u^*Y$. This shows the density, which implies that $C_k^*(Y)$ and $C_u^*Y \otimes \mathcal{K}(\ell^2(\mathbb{N}))$ are actually isomorphic. ☐

7. Finite propagation representatives

In this section, we prove that any class in a uniform K-homology group can be represented by a uniform Fredholm module with the operator having finite propagation. The proof follows the outline of the proof of analogous result in analytic K-homology.

Definition 7.1. An open cover of X is said to

- have *finite multiplicity*, if for any $R \geq 0$ there is $K \geq 0$, such that any ball with radius R intersects at most K elements of the cover;
- be *uniformly bounded*, if there is a common upper bound for all the diameters of members of the cover.

Remark 7.2. Any space X with bounded geometry admits uniformly bounded covers with finite multiplicity. However, bounded geometry alone produces such covers with possibly large bound on the diameters of the cover members. Consequently, a priori the propagation might not be made arbitrarily small (see the proof the next proposition). In order to achieve small propagation, we need some small scale (topological) assumption; for instance finite covering dimension would suffice.

Definition 7.3 (Finite propagation: continuous version)**.** Let H be a Hilbert space and let $\phi : C_0(X) \to \mathcal{B}(H)$ be a *-representation. We say that $T \in \mathcal{B}(H)$ has *finite propagation*, if there exists $R > 0$, such that $\phi(f)T\phi(g) = 0$ for every $f, g \in C_0(X)$ with $d(\mathrm{supp}(f), \mathrm{supp}(g)) \geq R$.

Proposition 7.4 (Uniform K-homology elements have representatives with finite propagation)**.** *Each uniform K-homology element over a space X with bounded geometry can be represented by a uniform Fredholm module (H, ϕ, S), where S is a finite propagation operator.*

Furthermore, we may assume that homotopies go through finite propagation operators as well.

Proof. Let (H, ϕ, T) be a uniform Fredholm module. Let $(U_i)_{i\in I}$ be a uniformly bounded open cover with finite multiplicity, and let $(\varphi_i^2)_{i\in I}$ be a continuous partition of unity subordinate to $(U_i)_{i\in I}$. By replacing the sets U_i by $N_\delta(U_i)$, the δ-neighborhoods for a fixed $\delta > 0$ and obtaining a partition of unity for the cover $(N_\delta(U_i))_i$, we can assume that all φ_i's are L_0-continuous for some $L_0 \geq 0$.

Denote $S = \sum_{i\in I} \varphi_i T \varphi_i$. This operator has finite propagation (which is bounded from above by $\sup_i \mathrm{diam}(U_i)$). We prove that (H, ϕ, S) is a uniform Fredholm module which represents the same uniform K-homology element as (H, ϕ, T).

Fix $\varepsilon > 0$ and $R, L > 0$. Let M be such that $[T, \phi(\cdot)]$ is $(\varepsilon, R, 2\max(L_0, L), M; \phi)$–approximable and that $T\phi(\cdot)$ and $\phi(\cdot)T$ are $(\varepsilon, R, M; \phi)$–approximable. Denote $S' = S - T = \sum_{i\in I} \varphi_i [T, \varphi_i]$. By finite multiplicity assumption, there is M_1, such that any ball with radius R intersects at most M_1 sets U_i. Take $f \in C_R(X)$ and consider $fS' = \sum_i f\varphi_i [T, \varphi_i]$. This sum has at most M_1 nonzero terms, and each of them is (ε, M)–approximable, hence fS' itself is $(M_1\varepsilon, MM_1)$–approximable. Similarly for $f \in C_{R,L}(X)$,

$$
\begin{aligned}
S'f &= \textstyle\sum_i \varphi_i [T, \varphi_i] f = \sum_i \varphi_i T \varphi_i f - \varphi_i^2 T f = \sum_i (\varphi_i T \varphi_i f - \varphi_i^2 f T) + \sum_i \varphi_i^2 [f, T] = \\
&= \textstyle\sum_i \varphi_i [T, f\varphi_i] + [f, T].
\end{aligned}
$$

The last term is (ε, M)–approximable by assumption, and again only at most M_1 terms in the sum are nonzero, and all of them are (ε, M)–approximable. Consequently, $S'f$ is $((M_1+1)\varepsilon, MM_1+1)$–approximable. Therefore we have proved that S' is uniform. Applying Lemma 2.16 finishes the first part of the proof.

For the part on homotopies, we just need to observe that the formula $\sum_{i\in I} \varphi_i T \varphi_i$ produces a continuous family if we vary T continuously, thanks to finite multiplicity of the chosen cover. ☐

8. Another picture of uniform Roe algebras

The definition of $C^*_k(Y)$ as given in section 6 inherently uses the standard basis of the auxiliary Hilbert space $\ell^2\mathbb{N}$. In this section, we develop a picture of $C^*_k(Y)$ starting with a general X-module (H,ϕ), instead of the concrete one ($\ell^2 Y\otimes\ell^2\mathbb{N}$, multiplication action). Furthermore, this model allows us to translate from "continuous" spaces X (which are needed in order to observe more than just 0–dimensional phenomena in (uniform) K-homology) to their discrete models $Y\subset X$ (which are supposed to be the targets of the index/assembly map).

Let us fix a metric space X for the rest of this section.

Definition 8.1 (Quasi-lattices, partitions)**.** We say that $Y\subset X$ is a *quasi–lattice*, if Y with induced metric is uniformly discrete space with bounded geometry, which is coarsely equivalent to X.

We say that a collection $(V_y)_{y\in Y}$ of subsets of X is a *quasi–latticing partition*, if each V_y is open, $V_x\cap V_y=\emptyset$ if $x\neq y$, $X=\bigcup_{y\in Y}\overline{V_y}$, $\sup_{y\in Y}\operatorname{diam}(V_y)<\infty$ and for every $\varepsilon>0$, $\sup_{y\in Y}|\{z\in Y\mid V_z\cap\operatorname{Nbhd}_\varepsilon(V_y)\neq\emptyset\}|<\infty$.

Remark 8.2. Not all spaces X have a quasi–lattice, but those with "bounded geometry" in any reasonable sense do. Furthermore, once there is a quasi–lattice, it's easy to produce quasi–latticing partitions (for instance by means of "pick the closest point in Y" map).

Example 8.3. A useful example to have in mind is the one of a graph X (with edges attached), with Y being its 0-skeleton. More generally, 0-skeleton of a uniformly locally finite simplicial polyhedron (endowed with the geodesic metric) is a quasi–lattice.

Recall that any *-homomorphism $\phi: C_0(X)\to\mathscr{B}(H)$ induces a Borel measure on X, and extends to a representation (also denoted by ϕ) of $\ell^\infty(X)$. We shall use this fact without mentioning explicitly throughout this section.

Definition 8.4 (Bases choice)**.** Given a metric space X, we define the *bases choice* $\mathscr{A}$ for X to be a 5-tuple $(Y,(V_y)_{y\in Y},H,\phi,\{\mathscr{S}_y\}_{y\in Y})$, where

- $Y\subset X$ is a quasi–lattice of X,
- $(V_y)_{y\in Y}$ is a quasi–latticing partition of X
- H is a Hilbert space, $\phi: C_0(X)\to\mathscr{B}(H)$ a non-degenerate $*$-representation[8],
- $\mathscr{S}_y=(e^y_i)_{i=1}^{N_y}$ is a basis of $H_y=\phi(\chi_{V_y})H$ (where we allow $N_y\in\mathbb{N}\cup\{\infty\}$ and we put by convention that $\mathscr{S}_y=\emptyset$ if $H_y=\{0\}$).

Such a bases choice determines a (possibly non-surjective) isometry $u_{\mathscr{A}}: H=\oplus_y H_y\to\ell^2(Y\times\mathbb{N})$.

Definition 8.5 (Realizations of $\mathscr{M}_k(C^*_uY\otimes\mathscr{K})$)**.** Let X be a metric space, $Y\subset X$ a quasi–lattice, $(V_y)_{y\in Y}$ a quasi–latticing partition, and let $\mathscr{A}_i=(Y,(V_y)_{y\in Y},H_i,\phi_i,\{\mathscr{S}^i_y\}_{y\in Y})$, $i=1,\dots,k$ be bases choices. Define the C*-algebra $C^*_k(X,\mathscr{A}_1,\dots,\mathscr{A}_k)\subset\mathscr{B}(\oplus_{i=1}^k H_i)$ as the closure of the algebra of the operators $T\in\mathscr{B}(\oplus_{i=1}^k H_i)$ satisfying the following conditions:

- T has finite propagation,
- there exists $M\geq 0$, such that each "entry" $T_{j,i;y,x}:\phi_i(\chi_{V_x})H_i\to\phi_j(\chi_{V_y})H_j$ only uses the first M basis vectors from bases $\mathscr{S}^i_x$, $\mathscr{S}^j_y$.

There is an injective *-homomorphism

$$\operatorname{Ad}(u_{\mathscr{A}_1}\oplus\cdots\oplus u_{\mathscr{A}_k}): C^*_k(X,\mathscr{A}_1,\dots,\mathscr{A}_k)\to\mathscr{M}_k(C^*_k(Y)).$$

We call the C*-algebra $C^*_k(X,\mathscr{A})\subset\mathscr{B}(H)$ the *$\mathscr{A}$-realization of $C^*_k(Y)$*.

Remark 8.6. Note that $C^*_k(X,\mathscr{A})$ is isomorphic only to a subalgebra of $C^*_k(Y)$ in general, but if each $\mathscr{S}_y$ is infinite, then $C^*_k(Y)$ and $C^*_k(X,\mathscr{A})$ are isomorphic.

Define $\operatorname{supp}(\mathscr{A})=\{y\in Y\mid\mathscr{S}_y\neq\emptyset\}$. If $\operatorname{supp}(\mathscr{A})$ is coarsely equivalent to Y, we have that $K_*(C^*_k(Y))\cong K_*(C^*_k(X,\mathscr{A}))$. More precisely, $C^*_k(Y)$ and $C^*_k(X,\mathscr{A})$ are Morita equivalent. Indeed, $\mathscr{M}_\infty(C^*_k(X,\mathscr{A}))\cong\mathscr{M}_\infty(C^*_u(\operatorname{supp}(\mathscr{A})))$, for Morita equivalence of $C^*_u(\operatorname{supp}(\mathscr{A}))$ and C^*_uY we refer to [5].

[8] A representation $\phi: C_0(X)\to\mathscr{B}(H)$ is non-degenerate, if $[\phi(C_0(X))]^\perp=\{0\}$

We continue by defining a relation between tuples of bases choices, in order to be able to get an inductive limit of realizations of $C_k^*(Y)$. We begin by a notion similar to an inclusion between a pair of bases choices.

Definition 8.7. Fix a quasi–lattice $Y \subset X$. Let $\mathscr{A}_i = \left(Y, (V_y^i)_{y\in Y}, H_i, \phi_i, \{\mathscr{S}_y^i\}\right)$, $i = 1,2$, be bases choices. We shall write $\mathscr{A}_1 \subseteq \mathscr{A}_2$, if the following conditions are satisfied:

- For each $y \in Y$, $\phi(\chi_{V_y^1})H_1$ is isometric to a subspace of $\phi(\chi_{V_y^2})H_2$ via an isometry v_y.
- Each v_y maps nth vector in the basis $\mathscr{S}_y^1$ to the n-vector in the basis $\mathscr{S}_y^2$.

A weaker version of $\subseteq$, denoted now by $\mathscr{A}_1 \sqsubseteq \mathscr{A}_2$, is defined in the same manner, except the last condition is replaced by

- for all $k \in \mathbb{N}$ there is $l \in \mathbb{N}$, such that for all $y \in Y$ the v_y-images of the first k vectors of $\mathscr{S}_y^1$ are among the linear span of the first l vectors of $\mathscr{S}_y^2$.

We now extend this inclusion to lists. Given two lists of bases choices $(\mathscr{A}_1,\dots,\mathscr{A}_k)$ and $(\mathscr{A}'_1,\dots,\mathscr{A}'_l)$ for X with respect to Y, we shall write $(\mathscr{A}'_1,\dots,\mathscr{A}'_l) \prec (\mathscr{A}_1,\dots,\mathscr{A}_k)$, if there is an injective function $\sigma : \{1,\dots,l\} \to \{1,\dots,k\}$, such that $\mathscr{A}'_i \subseteq \mathscr{A}_{\sigma(i)}$ for all $i = 1,\dots,l$. If this happens, then there is a natural embedding $i : C_k^*(X, \mathscr{A}'_1,\dots,\mathscr{A}'_l) \to C_k^*(X, \mathscr{A}_1,\dots,\mathscr{A}_k)$ (implemented by the *-homomorphism $\mathrm{Ad}(V)$, where $V = \oplus_y v_y$ is the isometric embedding of appropriate Hilbert spaces). This embedding commutes with maps between matrix algebras over $C_k^*(Y)$ as follows:

$$\begin{array}{ccc} C_k^*(X, \mathscr{A}'_1,\dots,\mathscr{A}'_l) & \longrightarrow & C_k^*(X, \mathscr{A}_1,\dots,\mathscr{A}_k) \\ \Big\downarrow{\scriptstyle \mathrm{Ad}(u_{\mathscr{A}'_1}\oplus\cdots\oplus u_{\mathscr{A}'_l})} & & \Big\downarrow{\scriptstyle \mathrm{Ad}(u_{\mathscr{A}_1}\oplus\cdots\oplus u_{\mathscr{A}_k})} \\ \mathscr{M}_l(C_k^*(Y)) & \xrightarrow{h_\sigma \otimes \mathrm{id}} & \mathscr{M}_k(C_k^*(Y)). \end{array} \tag{3}$$

By $h_\sigma : \mathscr{M}_l(\mathbb{C}) \to \mathscr{M}_k(\mathbb{C})$ we denote the embedding of matrix algebras determined by σ. More precisely, h_σ is the linear extension of the following assignment of matrix units $\mathscr{M}_l(\mathbb{C}) \ni e_{ij} \mapsto e_{\sigma(i)\sigma(j)} \in \mathscr{M}_k(\mathbb{C})$.

Furthermore, if we assume that $\mathscr{A}'_i = \mathscr{A}_{\sigma(i)}$ for $i = 1,\dots,l$, and if $\mathrm{supp}(\mathscr{A}_j)$ is coarsely equivalent to Y for each $j = 1,\dots,k$, then the top horizontal map induces an isomorphism on K-theory. This is a straightforward generalization of remark 8.6.

Note that for any bases choice $\mathscr{A} = (Y, (V_y)_{y\in Y}, H, \phi, \{\mathscr{S}_y\}_{y\in Y})$, there is another one $\mathscr{A}'$ with $\mathscr{A} \subseteq \mathscr{A}'$, such that $\mathrm{supp}(\mathscr{A}') = Y$. This can be arranged by choosing the Hilbert space of $\mathscr{A}'$ to be $H' = H \oplus \ell^2(Y \times \mathbb{N})$, the direct sum action of $C_0(X)$ and a suitable choice of bases $\mathscr{S}'_y$.

The previous discussion, together with Lemma 6.10, culminates in the following proposition:

Proposition 8.8 (A picture for $K_*(C_u^*Y)$)**.** *Let X be metric space and let $Y \subset X$ be a quasi–lattice. The collection $\mathscr{X}$ of all finite lists $(\mathscr{A}_1,\dots,\mathscr{A}_k)$ of bases choices for X with Y fixed forms a directed system. We have that there is an isomorphism η*

$$\eta : \lim_{\mathscr{X}} K_*(C_k^*(X, \mathscr{A}_1,\dots,\mathscr{A}_k)) \xrightarrow{\cong} K_*(C_u^*Y).$$

The following lemma shows that given a finite propagation uniform operator T on an X-module H, we can always find a bases choice $\mathscr{A}$, such that $T \in C_k^*(X, \mathscr{A})$.

Lemma 8.9. *Let X be metric space, let $Y \subset X$ be a quasi–lattice and let $(V_y)_{y\in Y}$ be a quasi–latticing partition of X. Let H be a Hilbert space and let $\phi : C_0(X) \to \mathscr{B}(H)$ be a *-homomorphism. Given a finite collection $T_1,\dots,T_k \in \mathscr{B}(H)$ of uniform operators with finite propagation, there exists a bases choice $\mathscr{A}$, such that $T_i \in C_k^*(X, \mathscr{A})$ for all $i = 1,\dots,k$.*

Proof. For simplicity, assume that we are given just one $T \in \mathscr{B}(H)$ to deal with (it will be clear that we can follow the procedure outlined below simultaneously for finitely many operators).

Denote $H_y = \phi(\chi_{V_y} H)$ and $T_{yz} = \phi(\chi_{V_y}) T \phi(\chi_{V_z}) \in \mathscr{B}(H_z, H_y)$. Since T has finite propagation and Y is uniformly discrete, there is a K, such that there are at most K nonzero entries in each column and row of the matrix $(T_{xz})_{x,z\in Y}$.

Fix $\varepsilon_1 = 1$ and take $R > \sup_{y\in Y} \mathrm{diam}(V_y)$. It follows from the assumption that there exists M, such that each T_{yz} is (ε_1, M)–approximable. Therefore, for each $y \in Y$, there are $2M$ orthonormal vectors $e_1^y,\dots,e_{2M}^y \in H_y$, for which there are $2M \times 2M$-matrices which in these (partial) bases represent operators $s_y \in \mathcal{B}(H_y)$ with $||T_{yy} - s_y|| < \varepsilon_1$.

Fix $y \in Y$ for a while and consider the "column" $(T_{yz})_{z \in Y}$. Each of them is (ε_1, M)–approximable, but not necessarily by a matrix in the so far chosen partial basis $e_1^y, \ldots, e_{2M}^y$. By adding at most M vectors to the chosen partial bases for H_y and H_z respectively, we can ensure that T_{yz} will be (ε_1, M)–approximable in the partial bases of H_y and H_z. We can do this for each nonzero T_{yz}, $z \in Y$, resulting in having chosen partial basis for H_y having at most $(2+K)M$ elements, and partial bases for H_z's having at most $3M$ elements. Doing this process for all $y \in Y$ results in choosing partial bases for each H_y having at most $(2+2K)M$ elements, now with the property that each T_{yz} is (ε_1, M)–approximable with matrices in the chosen partial bases.

To finish the construction, we choose a sequence of $\varepsilon_n > 0$ converging to 0 and do the above described process for each n, always just adding the newly chosen partial bases to the previous ones. Hence, we have constructed $\mathcal{A} = \mathcal{A}(Y)$. The fact that $T \in C_k^*(X, \mathcal{A})$ follows easily from the construction and the estimate 6.9. □

In fact, we can improve the previous lemma to finite collections of uniform operators which are do not necessarily have finite propagation, but are only approximable by finite propagation ones. To carry out the argument, we are going to use the relation $\sqsubseteq$ on bases choices (see definition 8.7). Note that if $\mathcal{A}_1 \sqsubseteq \mathcal{A}_2$, then $C_k^*(X, \mathcal{A}_1) \subset C_k^*(X, \mathcal{A}_2)$: Let $w \in C_k^*(X, \mathcal{A}_1)$ be finite propagation operator, such that a bound M on the number of basis vectors from $\mathcal{S}_y^1$ which are used in each entry w_{yz} of w. By the last condition in the definition of $\sqsubseteq$, there is a number M', such that for each $y \in Y$, the first M vectors of $\mathcal{S}_y^1$ are in the linear span of the first M' vectors of $\mathcal{S}_y^2$. Consequently, entries w_{yz} use only the first M' vectors of bases $\mathcal{S}_y^2$, and so $w \in C_k^*(X, \mathcal{A}_2)$.

Lemma 8.10. *Let X be metric space, let $Y \subset X$ be a quasi–lattice and let $(V_y)_{y \in Y}$ be a quasi–latticing partition of X. Let H be a Hilbert space and let $\phi : C_0(X) \to \mathcal{B}(H)$ be a *-homomorphism. Given a finite collection $T_1, \ldots, T_k$ in $\Theta(\phi)$, the C^*-algebra generated by uniform operators with finite propagation, there exists a bases choice $\mathcal{A}$, such that $T_i \in C_k^*(X, \mathcal{A})$ for all $i = 1, \ldots, k$.*

We isolate a part of the proof of the above lemma as another lemma, as it is useful by itself.

Lemma 8.11. *Let X be metric space, let $Y \subset X$ be a quasi–lattice and let $(V_y)_{y \in Y}$ be a quasi–latticing partition of X. Let H be a Hilbert space and let $\phi : C_0(X) \to \mathcal{B}(H)$ be a *-homomorphism. Assume that we are given a countable collection $\mathcal{A}_1, \ldots, \mathcal{A}_n, \ldots$ of bases choices of the form $(Y, (V_y)_{y \in Y}, H, \phi, \cdot)$. Then there exists a bases choice $\mathcal{A}$ of the same form, such that $\mathcal{A}_i \sqsubseteq \mathcal{A}$, $i \geq 1$.*

Proof. Denote $\mathcal{A}_n = (Y, (V_y)_{y \in Y}, H, \phi, \{\mathcal{S}_y^n\}_{y \in Y})$. We now define bases $\mathcal{S}_y$ out of $\mathcal{S}_y^n$ (and put $\mathcal{A} = (Y, (V_y)_{y \in Y}, H, \phi, \{\mathcal{S}_y\}_{y \in Y})$). Fix $y \in Y$ and enumerate the orthonormal bases $\mathcal{S}_y^n$ of the Hilbert space $\phi(\chi_{V_y})H$ as $(e_i^n)_{i \geq 1}$. We make one basis out of this sequence as follows: we fix a bijection $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ (for instance $\alpha(n, i) = \frac{1}{2}(n + i - 1)(n + i - 2) + i$; say we think of $\mathbb{N} \times \mathbb{N}$ to be the lattice points in the first quadrant of the plane, and we enumerate the points along the diagonals going from "top–left" to "right–bottom"). Let $(\beta_1, \beta_2) : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be its inverse. Now take the sequence of vectors $k \mapsto e_{\beta_2(k)}^{\beta_1(k)}$, and apply the Gramm–Schmidt orthogonalization process to it. We obtain a new basis $\mathcal{S}_y$, which obviously has the following property: for each $n \geq 1$ and $i \geq 1$, the vectors $e_i^n, \ldots, e_i^n$ are in the linear span of the first $\alpha(n, i)$ basis vectors of the new basis.

A quick glance at the definition of the relation $\sqsubseteq$ for bases choices shows that $\mathcal{A}$ is as required. □

Proof of Lemma 8.10. For simplicity, we concentrate on the case that $k = 1$, i.e. when we are given one operator $T \in \Theta(\phi)$. Note that T is uniform by the argument of Lemma 4.2. By assumption, T is approximable by a sequence T_n of uniform operators with finite propagation. For each T_n, there is a bases choice $\mathcal{A}_n = (Y, (V_y)_{y \in Y}, H, \phi, \{\mathcal{S}_y^n\}_{y \in Y})$, such that $T_n \in C_k^*(X, \mathcal{A}_n)$. Applying the previous lemma yields a bases choice $\mathcal{A}$, such that $\mathcal{A}_n \sqsubseteq \mathcal{A}$ for each $n \geq 1$. Since $C_k^*(X, \mathcal{A}_n) \subset C_k^*(X, \mathcal{A})$, T_n is a sequence of operators in $C_k^*(X, \mathcal{A})$ which converges to T. This finishes the proof. □

9. THE UNIFORM INDEX MAP

In the usual analytic K-homology, there is the index map (often also called the coarse assembly map) from the K-homology $K_*(X)$ of a space X to the K-theory of its Roe algebra $K_*(C^*X)$. But since Roe algebras of coarsely equivalent spaces are isomorphic, the target group of the index map can be understood as the K-theory $K_*(C^*Y)$ of the Roe algebra of any quasi–lattice $Y \subset X$.

The quickest way to define this map in the usual case is to use the reformulation of the K-homology as K-theory of a dual algebra (see [8, theorem 8.4.3] and section 4 for an analogous result in the uniform case) and then the 6-term exact sequence in K-theory, whose boundary maps become the assembly maps. For details of this construction, see for instance [8, section 12.3].

The goal of this section is to construct a similar index/assembly map in the uniform setting. More precisely, we define a homomorphism $\mu_u : K_*^u(X) \to K_*(C_u^*Y)$ for a quasi–lattice $Y \subset X$ in a metric space X. However, instead of the C*-algebra route, we take a more hands-on approach.

In this paragraph, we recall a formula for the usual assembly map. If (H,ϕ,S) is a 0-Fredholm module, we can define its index as follows: denote

$$W = \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -S^* & 1 \end{pmatrix} \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathcal{B}(H)).$$

This is an invertible in $\mathcal{M}_2(\mathcal{B}(H))$. Then put $\operatorname{ind}(S) = W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^{-1} \in \mathcal{M}_2(\mathcal{B}(H))$. Concretely,

$$\operatorname{ind}(S) = \begin{pmatrix} SS^* + (1-SS^*)SS^* & S(1-S^*S) + (1-SS^*)S(1-S^*S) \\ S^*(1-SS^*) & (1-S^*S)^2 \end{pmatrix}.$$

A simple computation shows that $\operatorname{ind}(S)$ is actually an idempotent in $\mathcal{M}_2(\mathcal{B}(H))$. Furthermore, $\partial(H,\phi,S) = [\operatorname{ind}(S)] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]$ is a K_0-class in the K-theory group of appropriate algebra, modulo which is S invertible. For example, starting with a finite propagation S, one gets $\partial(H,\phi,S)$ in $K_0(C^*X)$, the K-theory of the Roe C*-algebra.

Starting with a 1-Fredholm module (H,ϕ,Q), its index can be constructed using the formula $\operatorname{ind}(Q) = \exp(-2\pi i \frac{Q+1}{2}) \in \mathcal{B}(H)$. The operator $\operatorname{ind}(Q)$ is invertible[9], but even if we start with a finite propagation Q, $\operatorname{ind}(Q)$ might not have finite propagation. However, it is approximable by finite propagation invertibles in this case, hence still gives a class $[\operatorname{ind}(Q)] \in K_1(C^*X)$.

Let us now turn to the uniform case. Fix a quasi–lattice $Y \subset X$. We define $\mu_u : K_*^u(X) \to K_*(C_u^*Y)$ in the following proposition:

Proposition 9.1 (Uniform index map, even case). *Let (H,ϕ,S) be a 0-uniform Fredholm module with S having finite propagation and ϕ being non-degenerate. For any quasi–lattice $Y \subset X$, there exists a bases choice $\mathcal{A} = (Y,(V_y)_{y\in Y},H,\phi,\{(e_i^y)_{i\in\mathbb{N}}\}_{y\in Y})$, such that $\operatorname{ind}(S) \in \mathcal{M}_2(\mathcal{B}(H))$ is an idempotent that actually belongs to $C_k^*(X,\mathcal{A},\mathcal{A})$. Furthermore, we can define a group homomorphism $\mu_u : K_0^u(X) \to K_0(C_u^*Y)$ by*

$$\mu_u[(H,\phi,S)] = \eta_*([\operatorname{ind}(S)] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]) \in K_0(C_u^*Y),$$

i.e. the right-hand side does not depend on the choices made. Recall that η is described in proposition 8.8.

Proposition 9.2 (Uniform index map, odd case). *Let (H,ϕ,Q) be a 1-uniform Fredholm module with Q having finite propagation and ϕ being non-degenerate. For any quasi–lattice $Y \subset X$ there exists a bases choice $\mathcal{A} = (Y,(V_y)_{y\in Y},H,\phi,\{(e_i^y)_{i\in\mathbb{N}}\}_{y\in Y})$, such that $\operatorname{ind}(Q) \in \mathcal{B}(H)$ is an invertible that actually belongs to $C_k^*(X,\mathcal{A})^+$. Furthermore, the map $\mu_u : K_1^u(X) \to K_1(C_u^*Y)$ defined by*

$$\mu_u[H,\phi,Q] = \eta_*[\operatorname{ind}(Q)] \in K_1(C_u^*Y)$$

is a group homomorphism.

Proof of the 0-case. Picking any quasi–latticing partition $(V_y)_{y\in Y}$, the existence of a suitable $\mathcal{A}$ follows from Lemma 8.9, applied to the four entries of $\operatorname{ind}(S)$, which are uniform and have finite propagation.

It is clear that our construction of the index preserves direct sums. Also, the index of a degenerate element gives zero in the K-theory. Indeed, if (H,ϕ,S) is a degenerate 0-Fredholm module, then $\phi(f)\operatorname{ind}(S) = \begin{pmatrix} \phi(f) & 0 \\ 0 & 0 \end{pmatrix}$ for any $f \in C_0(X)$, so by using a partition of unity we obtain that $\operatorname{ind}(S) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Thus, to finish the proof, we need to show the independence of the index on the choice of $\mathcal{A}$, and under homotopies of uniform Fredholm modules. Our proof for homotopies includes the argument for choices of

[9] When we talk about invertibles in a non-unital C*-algebra, we mean that they are invertible in the unitization.

$\mathscr{A}$, since we can just take a constant homotopy, and choose different bases choices at the endpoints. We shall now outline the proof for homotopies.

Assume that we are given a homotopy (H, ϕ_t, S_t) of uniform Fredholm modules. We assume that all S_t have finite propagation (see proposition 7.4), so that the index as we have defined it can be constructed. Note that the requirements on ϕ_t ensure that $B = \Theta(\phi_t)$, the C*-algebra generated by all ϕ-uniform operators with ϕ-finite propagation, does not depend on t.

By applying the index formula to S_t, we obtain a norm-continuous path of projections in $\mathscr{M}_2(B) \subset \mathscr{M}_2(\mathscr{B}(H))$. For the sake of simplicity, let us assume that we have a norm-continuous path of projections (T_t) in B itself.

Choose $\mathscr{A}_0$ and $\mathscr{A}_1$ to be bases choices corresponding to (H, ϕ_0) and (H, ϕ_1) respectively, such that $T_i \in C^*_k(X, \mathscr{A}_i)$, $i = 0, 1$. Now we are in the position to apply the following lemma 9.3, which finishes the proof for the even case. □

Lemma 9.3. *Let H be a Hilbert space, $\phi : C_0(X) \to \mathscr{B}(H)$ a *-representation. Denote $B = \Theta(\phi) \subset \mathscr{B}(H)$, the C*-algebra generated by ϕ-uniform operators with ϕ-finite propagation. Assume that T_t, $t \in [0,1]$ is a homotopy of projections in B, and that $\mathscr{A}_0$ and $\mathscr{A}_1$ are two bases choices, such that $T_i \in C^*_k(X, \mathscr{A}_i)$[10]. Then $[T_0] = [T_1] \in K_0(C^*_k(X, \mathscr{A}_0, \mathscr{A}_1))$.*

Proof. Since T_t is a homotopy of projections in a C*-algebra B, there exists an invertible element $v_0 \in B$ with $||v_0|| = 1$, such that $T_1 = v_0^{-1} T_0 v_0$ (see e.g. [2, Proposition 4.3.2]). Note that v_0 might not have finite propagation, so we will need to make some approximations further on.

The images of T_0 and T_1 under the inclusions of $C^*_k(X, \mathscr{A}_0)$ and $C^*_k(X, \mathscr{A}_1)$ into $C^*_k(X, \mathscr{A}_0, \mathscr{A}_1) \subset \mathscr{M}_2(\mathscr{B}(H))$ are the operators $\left(\begin{smallmatrix} T_0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & T_1 \end{smallmatrix}\right)$. These two projections are Murray–von Neumann equivalent by the elements $x = \left(\begin{smallmatrix} 0 & T_0 v_0 \\ 0 & 0 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 0 & 0 \\ v_0^{-1} T_0 & 0 \end{smallmatrix}\right)$. To finish the argument, we must show that $x, y \in C^*_k(X, \mathscr{A}_0, \mathscr{A}_1)$.

For the rest of the proof, we will think of $\mathscr{M}_k = \mathscr{M}_k(\mathbb{C})$ as $\mathscr{B}(\mathrm{span}(e_1, \ldots, e_k))$ in $\mathscr{K}(\ell^2\mathbb{N})$, where $e_1, e_2, \ldots$ is the standard basis of $\ell^2\mathbb{N}$. Let $A \subset \mathscr{B}(\ell^2(Y) \otimes \ell^2\mathbb{N})$ be the algebra of all finite propagation matrices $(t_{yz})_{y,z\in Y}$ for which there exists $k \in \mathbb{N}$ with $t_{yz} \in \mathscr{M}_k$ for all $y, z \in Y$. Then $C^*_k(Y)$ is the norm closure of A.

We shall give a proof that $y \in C^*_k(X, \mathscr{A}_0, \mathscr{A}_1)$; a proof for x is analogous. Denote $u_0 = u_{\mathscr{A}_0}$ and $u_1 = u_{\mathscr{A}_1}$.

We need to show that $y \in \mathrm{Ad}(u_0 \oplus u_1)(\mathscr{M}_2(C^*_k(Y)))$. This will follow from the following statement: For any $\varepsilon > 0$, there exists $p \in A$, such that $||p - u_1 v_0^{-1} T_0 u_0^*|| < \varepsilon$. By the choice of $\mathscr{A}_0$ and $\mathscr{A}_1$, we know that there are $\hat{s}_0, \hat{s}_1 \in \mathscr{B}(H)$, such that $u_0 \hat{s}_0 u_0^*, u_1 \hat{s}_1 u_1^* \in A$, $||\hat{s}_0 - T_0|| < \varepsilon$ and $||\hat{s}_1 - v_0^{-1} T_0 v_0|| < \varepsilon$. Note that $\hat{s}_0$ and $\hat{s}_1$ have finite propagation. Furthermore, there exists an invertible element $v \in B$ with finite propagation, norm 1, and $||v - v_0|| < \varepsilon$ and $||v^{-1} - v_0^{-1}|| < \varepsilon$. It follows that $||v \hat{s}_1 v^{-1} - T_0|| < 3\varepsilon$.

At this moment, the setting is as follows: we have finite propagation operators $v, \hat{s}_0, \hat{s}_1$ and T_0, such that $||\hat{s}_0 - T_0|| < \varepsilon$, $||v \hat{s}_1 v^{-1} - T_0|| < 3\varepsilon$.

Claim 4. There exists $p \in A$, such that $||p - u_1 \hat{s}_1 v^{-1} u_0^*|| < 4\varepsilon$.

Proof of claim. Combining the two inequalities with T_0 gives

$$4\varepsilon > ||\hat{s}_0 - v \hat{s}_1 v^{-1}|| = ||v^{-1} \hat{s}_0 - \hat{s}_1 v^{-1}|| \geq ||u_1 v^{-1} u_0^* u_0 \hat{s}_0 u_0^* - u_1 \hat{s}_1 u_1^* u_1 v^{-1} u_0^*||$$

Denoting $w = u_1 v^{-1} u_0^* \in \mathscr{B}(\ell^2(Y) \otimes \ell^2(\mathbb{N}))$, $s_0 = u_0 \hat{s}_0 u_0^* \in A$, $s_1 = u_1 \hat{s}_1 u_1^* \in A$, we obtain $||w s_0 - s_1 w|| < 4\varepsilon$. Note that w has finite propagation. Let k be such that all entries s_0 and s_1 belong to $\mathscr{M}_k$. We split the standard basis of $\ell^2(Y) \otimes \ell^2\mathbb{N}$ into two sets $\mathscr{B}_1$ (first k vectors from each $\{y\} \otimes \ell^2\mathbb{N}$) and $\mathscr{B}_2$ (the other basis vectors). With respect to this decomposition, we can write $s_0 = \left(\begin{smallmatrix} \star & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $s_1 = \left(\begin{smallmatrix} s_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} w_{11} & w_{12} \\ \star & \star \end{smallmatrix}\right)$. Consequently,

$$4\varepsilon > ||w s_0 - s_1 w|| = \left\| \left(\begin{smallmatrix} \star & 0 \\ \star & 0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} s_{11} w_{11} & s_{11} w_{12} \\ 0 & 0 \end{smallmatrix}\right) \right\| = \left\| \left(\begin{smallmatrix} \star & s_{11} w_{12} \\ \star & 0 \end{smallmatrix}\right) \right\|.$$

Hence $||s_{11} w_{12}|| < 4\varepsilon$. Denoting $p = \left(\begin{smallmatrix} s_{11} w_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, we immediately see that $p \in A$ and $||s_1 w - p|| = \left\| \left(\begin{smallmatrix} 0 & s_{11} w_{12} \\ 0 & 0 \end{smallmatrix}\right) \right\| < 4\varepsilon$. □

[10]For any bases choice $\mathscr{A}$, $C^*_k(X, \mathscr{A}) \subset B$. The uniformity of $T \in C^*_k(X, \mathscr{A})$ follows the formula $f = \sum_y f \chi_{V_y}$. Note that for fixed $R \geq 0$ and $f \in C_R(X)$, there is a uniform bound on the number of nonzero terms in the sum by bounded geometry.

Returning to the proof of the lemma, we conclude

$$||p - u_1 v_0^{-1} T_0 u_0^*|| \le ||p - u_1 v^{-1} T_0 u_0^*|| + \varepsilon ||T_0|| \le$$
$$\le ||p - u_1 \hat{s}_1 v^{-1} u_0^*|| + \varepsilon ||T_0|| + 3\varepsilon < 4C\varepsilon + \varepsilon ||T_0|| + 3\varepsilon.$$

This finishes the proof. □

Proof of the 1*-case.* The operator $\operatorname{ind}(Q) = \exp(-2\pi i \frac{Q+1}{2}) - 1 \in \mathscr{B}(H)$ is uniform ($P = \frac{Q+1}{2}$ satisfies $P^2 \sim_{ua} P$ and so $\exp(-2\pi i P) - 1 \sim_{ua} P(\exp(-2\pi i) - 1) = 0$), but might not have finite propagation. However, from the formula for $\operatorname{ind}(Q)$ and finite propagation of Q it follows that $\operatorname{ind}(Q) - 1 \in \Theta(\phi)$, and so the existence of suitable $\mathscr{A}$ follows from Lemma 8.10 (after we have fixed some quasi-latticing partition $(V_y)_{y \in Y}$).

We reduce the independence of the index on homotopies to independence on bases choices. Taking a homotopy (H, ϕ_t, Q_t) of 1-uniform Fredholm modules, we assume that all Q_t have finite propagation. It follows that $U_t = \operatorname{ind}(Q_t)$, $t \in [0,1]$ is a homotopy of invertibles in $B^+ = \Theta(\phi_0)^+$. Since the set of invertibles is open, by a standard compactness argument we can assume that the homotopy is piecewise-linear. Hence, it is sufficient to assume that we have just one linear path of invertibles from (say) U_0 to U_1 in B^+, and that we are given two bases choices $\mathscr{A}_0$ and $\mathscr{A}_1$, such that $U_i \in C_k^*(X, \mathscr{A}_i)^+$, $i = 1,2$. Applying Lemma 8.11 gives a bases choice $\mathscr{A}$, such that $\mathscr{A}_i \sqsubseteq \mathscr{A}$ for each $i = 1,2$. Hence U_0, U_1 and the whole (linear) homotopy between them is actually in $C_k^*(X, \mathscr{A})^+$. So $[U_0] = [U_1] \in K_1(C_k^*(X, \mathscr{A}))$, and the assertion will follow from the independence of the index on the choice of a bases choice.

We find ourselves in the following situation: we are given an invertible $U = 1 + K$, $K \in B = \Theta(\phi)$, and two bases choices $\mathscr{A}_0$, $\mathscr{A}_1$, such that $K \in C_k^*(X, \mathscr{A}_i)$, $i = 0,1$.

We will think of $\mathscr{M}_k = \mathscr{M}_k(\mathbb{C})$ as $\mathscr{B}(\operatorname{span}(e_1, \ldots, e_k))$ in $\mathscr{K}(\ell^2\mathbb{N})$, where $e_1, e_2, \ldots$ is the standard basis of $\ell^2\mathbb{N}$. Let $A \subset \mathscr{B}(\ell^2(Y) \otimes \ell^2\mathbb{N})$ be the algebra of all finite propagation matrices $(t_{yz})_{y,z \in Y}$ for which there exists $k \in \mathbb{N}$ with $t_{yz} \in \mathscr{M}_k$ for all $y, z \in Y$. Then $C_k^*(Y)$ is the norm closure of A. Denote $u_0 = u_{\mathscr{A}_0}$ and $u_1 = u_{\mathscr{A}_1}$.

We will prove that $\left(\begin{smallmatrix} U & 0 \\ 0 & 1 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & U \end{smallmatrix}\right) \in C_k^*(X, \mathscr{A}_0, \mathscr{A}_1)^+$. The standard rotation homotopy between these two matrices has the form $1 + \begin{pmatrix} \sin^2(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) \\ \cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) & \cos^2(\frac{\pi}{2}t) \end{pmatrix} K$, and so it is sufficient to prove that actually $\left(\begin{smallmatrix} 0 & K \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ K & 0 \end{smallmatrix}\right) \in C_k^*(X, \mathscr{A}_0, \mathscr{A}_1)$. Equivalently, that $u_0 K u_1^*$ and $u_1 K u_0^* \in A$.

Pick $\varepsilon > 0$. Since $K \in C_k^*(X, \mathscr{A}_i)$, $i = 0,1$, there exist $\hat{s}_0, \hat{s}_1 \in \mathscr{B}(H)$ with finite propagation, such that $s_i := u_i \hat{s}_i u_i^* \in A$ and $||\hat{s}_i - K|| < \varepsilon$ for $i = 0,1$. Since $K \in B$, there exist an operator $\hat{K} \in B$ with finite propagation, such that $||K - \hat{K}|| < \varepsilon$, $i = 0,1$. Consequently, $||\hat{s}_i - \hat{K}|| < 2\varepsilon$ for $i = 0,1$.

At this moment, we can apply the proof of Claim 4 above (with $v = 1$ and $T_0 = K$, otherwise verbatim), to obtain $p \in A$, such that $||p - u_1 K u_0^*|| < 8\varepsilon$. Letting $\varepsilon \to 0$, we obtain that $u_1 K u_0^* \in A$. Analogous proof shows also $u_0 K u_1^* \in A$. We are done. □

10. On the Baum–Connes Conjecture with ℓ^∞-coefficients

As an application of the Mayer–Vietoris sequence for uniform K-homology, we exhibit a connection with the Baum–Connes conjecture.

Yu [14] proved that for a discrete group Γ, the Baum–Connes conjecture [1] for Γ with coefficients in $\ell^\infty(\Gamma, \mathscr{K})$ is equivalent to the Coarse Baum–Connes conjecture for Γ. The right-hand side in both conjectures is the K-theory of the Roe C*-algebra $C^*\Gamma \cong \ell^\infty(\Gamma, \mathscr{K}) \rtimes_r \Gamma$. The core of Yu's proof is showing the left-hand sides are the same, i.e. that

$$K_*^{\mathrm{top}}(\Gamma, \ell^\infty(\Gamma, \mathscr{K})) = \lim_{\substack{\Delta \subset \underline{B}\Gamma, \\ \Delta \text{ compact}}} KK_*^\Gamma(C_0(\rho^{-1}(\Delta)), \ell^\infty(\Gamma, \mathscr{K})) \cong \lim_{d \to \infty} K_*(P_d\Gamma),$$

where $\rho : \underline{E}\Gamma \to \underline{B}\Gamma$ denotes the quotient map. We prove an analogous statement for uniform K-homology in certain cases:

Theorem 10.1. *If* Γ *is a torsion-free countable discrete group, then*

$$\lim_{\substack{\Delta \subset \underline{B}\Gamma, \\ \Delta \text{ compact}}} KK_*^\Gamma(C_0(\rho^{-1}(\Delta)), \ell^\infty\Gamma) \cong \lim_{d \to \infty} K_*^u(P_d\Gamma),$$

As a consequence, this provides a computation of $\lim_{d\to\infty} K^u_*(P_d\Gamma)$ for torsion-free discrete groups for which the Baum–Connes conjecture with commutative coefficients is known, for instance $\mathbb{Z}^n$ or the free groups (see e.g. [7, 10]).

We consider a discrete group Γ endowed with a proper, left-invariant metric. Such a metric makes Γ into a uniformly discrete space with bounded geometry. There are such metrics on every discrete group, and any two such are quasi-isometric. For instance, if Γ is finitely generated, the word metric provides an example of such a metric.

Proof of 10.1. First, realize that for countable discrete groups

$$\lim_{\substack{\Delta\subset \underline{B}\Gamma,\\ \Delta \text{ compact}}} KK^\Gamma_*(C_0(\rho^{-1}(\Delta)),\ell^\infty\Gamma) \cong \lim_{d\to\infty} KK^\Gamma_*(C_0(P_d\Gamma),\ell^\infty\Gamma).$$

The rest of the proof is devoted to showing that the right-hand side above is in fact isomorphic to $\lim_{d\to\infty} K^u_*(P_d\Gamma)$, by using the Mayer–Vietoris sequence.

We proceed similarly as in [14, proof of Theorem 2.7]. Let X be a Γ-invariant subset of $\rho^{-1}(\Delta)$, where $\Delta\subset \underline{B}\Gamma$ is compact. We construct a homomorphism

$$\psi : K^u_*(X) \to KK^\Gamma_*(C_0(X),\ell^\infty\Gamma)$$

as follows: given a uniform Fredholm module (H,ϕ,F) for X, we let $H' = H\otimes \ell^\infty\Gamma \cong \ell^\infty(\Gamma,H)$, a Hilbert module over $\ell^\infty\Gamma$. The group Γ acts on it by translations. Furthermore, we define $\phi' : C_0(X)\to \mathscr{B}(H')$ by

$$(\phi'(f)\xi)(\gamma) = (\phi(\gamma^* f))\xi(\gamma)$$

for $f\in C_0(X)$, $\xi\in H'\cong \ell^\infty(\Gamma,H)$, $\gamma\in\Gamma$, and where γ^* denotes the action of γ on $C_0(X)$. Finally, we put $F'\in\mathscr{B}(H')$ to be the operator given by $(F'\xi)(\gamma) = F(\xi(\gamma))$. It is straightforward to check that the triple (H',ϕ',F') is a Fredholm Γ-module. Since Γ acts on X by isometries, note that $\gamma^* f$ has the same support and the same L-continuity as f, and so the size of matrices which approximate expressions like $(F^2-1)\phi(\gamma^* f)$ does not depend on γ, only on f. We let the image of $[(H,\phi,F)]$ under ψ to be $[(H',\phi',F')]$. It is immediate that this assignment is well-defined, and it describes a group homomorphism. We prove that ψ is an isomorphism for $X = P_d\Gamma$.

For $X = P_d\Gamma$, there exists a finite cover $\{U_i\}_{i=1}^m$ of X, such that each U_i is a space of the form $\Gamma\times Y$, where Y is contractible, compact, with the diameter at most $\frac{1}{2}$. Such a cover can be constructed by considering sufficiently fine barycentric subdivision of the finite simplicial complex $P_d\Gamma/\Gamma$, and then pulling it back to $P_d\Gamma$ by ρ.

We now use the Mayer–Vietoris sequences for both KK^Γ_* and K^u_* (Theorem 5.1) simultaneously for showing that ψ is an isomorphism for $X = P_d\Gamma$. Note that ψ commutes with all the involved Mayer–Vietoris sequences. This, together with an induction process, reduces the general case to the case when $X = \Gamma\times Y$, Y is as above. By [14, Lemma 2.3], $KK^\Gamma_*(C_0(\Gamma\times Y),\ell^\infty\Gamma)$ can be identified with $KK_*(C_0(Y),\ell^\infty\Gamma)$. Under this identification, the map

$$\psi : K^u_*(\Gamma\times Y) \to KK_*(C_0(Y),\ell^\infty\Gamma)$$

can be understood as follows: Given a uniform Fredholm module (H,ϕ,F) for $\Gamma\times Y$, denote $H_\gamma = \phi(\chi_{\{\gamma\}\times Y})H$. Then $H' = \oplus_{\gamma\in\Gamma} H_\gamma$ is naturally a Hilbert module over $\ell^\infty\Gamma$ (with the coordinate-wise inner product). Furthermore, we let $\phi' : C_0(Y)\to\mathscr{B}(H')$ be defined as $((\phi'(f))\xi)(\gamma) = \phi(\chi_{\{\gamma\}\times Y}\cdot f)(\xi(\gamma))$ for $f\in C_0(Y)$, $\xi\in H'$, $\gamma\in\Gamma$. Finally, define $F'\in\mathscr{B}(H')$ by $F' = \oplus_{\gamma\in\Gamma}\phi(\chi_{\{\gamma\}\times Y})F\phi(\chi_{\{\gamma\}\times Y})$. With this notation, ψ assigns the Fredholm module $[(H',\phi',F')]$ to $[(H,\phi,F)]$. Now it is easy to see that ψ is an isomorphism, since we may assume that F has propagation at most $\frac{1}{2}$, by Proposition 7.4, remark 7.2 and the properties of the space $\Gamma\times Y$. This concludes the proof. □

Remark 10.2. From the constructions in the above proof, the isomorphism commutes with the uniform index map. Moreover, from the definition of the index map it is clear that in fact it coincides with the Baum–Connes map with coefficients in $\ell^\infty\Gamma$ when Γ is torsion-free.

Corollary 10.3. *The statement that*

$$\mu_{\mathrm{u}} : \lim_{d\to\infty} K^u_*(P_d\Gamma) \to K_*(C^*_u\Gamma)$$

is an isomorphism for a torsion-free countable discrete group Γ (an analogue of the Coarse Baum–Connes conjecture) is equivalent to the Baum–Connes conjecture for Γ with coefficients in $\ell^\infty\Gamma$.

Remark 10.4. It is likely that the conclusion of Theorem 10.1 holds without any assumption on torsion. However, that would require at least some degree of homotopy invariance of the uniform K-homology, which would allow us to pass from $\underline{E}\Gamma$ to $E\Gamma$ on the K-homology side (cf. [14, Lemma 2.10]).

11. Amenability

As an application of uniform K-homology, we prove a criterion for amenability. It is analogous to similar criteria in the context of uniformly finite homology [3] and K-theory of uniform Roe algebras [6]. Both of these can be interpreted as saying that a space (which is uniformly discrete and has bounded geometry) is amenable if and only if its "fundamental class" is nontrivial in the appropriate group. In [3], it is the usual fundamental class in the 0th uniformly finite homology group; in [6] it is the class of the identity operator [1] in the K_0 group of the uniform Roe algebra. Our criterion (Theorem 11.2) has the same form.

Recall (Følner's) definition of amenability (see [3, section 3]).

Definition 11.1. Let Y be a uniformly discrete metric space. For a set $U \subset Y$, we define its *r-boundary* by

$$\partial_r U = \{y \in Y \mid d(y,U) \leq r \text{ and } d(y, Y \setminus U) \leq r\}.$$

We say that Y is *amenable*, if for any $r, \delta > 0$, there exists a finite set $U \subset Y$, such that

$$\frac{|\partial_r U|}{|U|} < \delta.$$

Note that this definition is equivalent to the usual definition of amenability of groups (existence of an invariant mean) for spaces arising as Cayley graphs of discrete groups. However, we do not require the Følner sets to exhaust the whole space, and so we need to be cautious when applying this to general metric spaces. For instance, taking any uniformly discrete metric space Y, one can make it amenable by attaching an infinite "spaghetti" to it, i.e. an infinite ray. Also note that any "coarse disjoint union finite spaces" is also amenable in this sense, since for a given $r > 0$, we can always select a finite piece U of the space, which is at least r-far from the rest of the space, hence making $\partial_r U = \emptyset$. In particular, this applies to expanders.

Let X be a graph (with the edges attached) and let Y be its vertex set. Recall the definition of the fundamental class $\mathbf{S} \in K_0^u(X)$ (see example 2.9). Let $H = \ell^2 Y \otimes \ell^2\mathbb{N}$, and endow H with the multiplication action of $C_0(X)$. Let $S \in \mathscr{B}(\ell^2\mathbb{N})$ be the unilateral shift. Let $\tilde{S} = \operatorname{diag}(S) \in \mathscr{B}(H)$ and finally denote $\mathbf{S} = [(H, \phi, \tilde{S})]$. It is easy to see that $\mathbf{S} \in K_0^u(X)$, and that $\operatorname{ind}(\tilde{S}) = 1 \otimes p_0 \in \mathscr{B}(\ell^2 Y \otimes \ell^2\mathbb{N})$, where p_0 is a rank one projection (onto $\mathbb{C}e_1 \in \ell^2\mathbb{N}$). We also denote by $0 \in K_0^u(Y)$ the trivial element.

Theorem 11.2. *Let X be a connected graph with the vertex set Y. Then Y is amenable if and only if $\mathbf{S} \neq 0$ in $K_0^u(X)$.*
More generally, if X is not connected, then Y is amenable if and only if there exists $C \geq 0$, such that $\mathbf{S} \neq 0$ in $K_0^u(P_C(Y))$ (recall that $P_C(Y)$ denotes the Rips complex of Y, see definition 6.3).

Remark 11.3. Note that the technical assumption that Y is a graph is not too restrictive, since every metric space with bounded geometry is coarsely equivalent to a graph.

Proof. If Y is amenable, then $\mu_u(\mathbf{S}) = [1] \neq [0] = \mu_u(0) \in K_0(C_u^*Y)$ by [6], and so $\mathbf{S} \neq 0$. For the convenience of the reader, let us sketch this part of Elek's proof. The idea is that if Y is amenable, then using Følner sets B_n, one can construct a trace on C_u^*Y as an ultralimit of functions $f_n(T) = \frac{1}{|B_n|}\sum_{x \in B_n} t_{xx}$. Trace then distinguishes [1] from [0] in $K_0(C_u^*Y)$.

Let us turn to the reverse implication. Assume that Y is not amenable. We will proceed to constructing a homotopy connecting $\mathbf{S}$ and 0 in $K_0^u(X)$.

First, we describe a "building block". Denote $I = [0,1]$. Denote $T_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & S \end{smallmatrix}\right)$ and $T_1 = \left(\begin{smallmatrix} S & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathscr{B}(H_I)$, where $H_I = \ell^2\mathbb{N} \oplus \ell^2\mathbb{N}$. Let the action ψ of $C(I)$ on H_I be $\psi(f)(\eta \oplus \xi) = f(0)\eta \oplus f(1)\xi$. Let us show a homotopy (H_I, ψ_t, T_t) between (H_I, ψ, T_0) and (H_I, ψ, T_1). Define

$$\psi_t(f)(\eta \oplus \xi) = \begin{cases} f(0)\eta \oplus f(1-3t)\xi & 0 \leq t \leq \frac{1}{3} \\ f(0)\eta \oplus f(0)\xi & \frac{1}{3} \leq t \leq \frac{2}{3} \\ f(0)\eta \oplus f(3t-2)\xi & \frac{2}{3} \leq 1 \end{cases}$$

and

$$T_t = \begin{cases} T_0 & 0 \le t \le \frac{1}{3} \\ \alpha_t \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} \alpha_t^* & \frac{1}{3} \le t \le \frac{2}{3} \\ T_1 & \frac{2}{3} \le 1, \end{cases}$$

where $\alpha_t = \begin{pmatrix} \cos(\frac{\pi}{2}(3t-1)) & \sin(\frac{\pi}{2}(3t-1)) \\ -\sin(\frac{\pi}{2}(3t-1)) & \cos(\frac{\pi}{2}(3t-1)) \end{pmatrix}$ is the rotation homotopy. It is clear that operators $\begin{pmatrix} S^k & 0 \\ 0 & S^l \end{pmatrix}$ and $\begin{pmatrix} S^{k-1} & 0 \\ 0 & S^{l+1} \end{pmatrix}$ (on the same Hilbert space with the same action of $C(I)$) are homotopic as well.

Now we turn to $Y \subset X$. Assuming non-amenability of Y and applying [3, Theorem 3.1 and Lemma 2.4], for each $y \in Y$ there exists a "tail", i.e. a sequence $(z_i^y)_{i\ge 0} \subset Y$, such that $z_0 = y$, $C = \sup_{y,i}(d(z_i^y, z_{i+1}^y)) < \infty$, satisfying the condition that in every ball of a fixed radius, the number of tails passing through is uniformly bounded.

In the case when X is connected, we can reduce the general C to the case $C = 1$, i.e. to the situation when the tails actually follow the edges of X. We may achieve this just by refining the tails, without violating the condition on uniform bound on tails passing through balls, since Y has bounded geometry.

If we do not assume connectedness, we may get by working with the Rips complex $P_C(Y)$ instead of $X = P_1(Y)$, since any two points with distance $\le C$ are connected by an edge in $P_C(Y)$.

Consequently, it is possible to partition the collection of edges contained in all tails $((z_i^y, z_{i+1}^y))_{y\in Y, i\in\mathbb{N}}$ (we allow for multiplicities) into finitely many parts $A_1, \ldots, A_k$, such that no two edges from the same part share a common vertex.

The idea of the rest of the construction is to "send off" the $\tilde{S}$ along the tails off to infinity, and thus connecting $\tilde{S}$ with 1. This is done in k steps. In step j, we simultaneously apply the building block construction to each of the edges in A_j (this is possible by the choice of A_j), thus "transferring" one S along each of those edges. After each step, we obtain a diagonal matrix in $\mathscr{B}(H)$ with various powers of S on the diagonal. The whole homotopy begins with $\tilde{S}$, and ends with 1, since after all k steps the S from each $y \in Y$ was shifted away from y along the tail. □

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