ANOTHER TAKE ON COARSE EMBEDDINGS OF HYPERBOLIC GROUPS

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ABSTRACT. We give yet another proof of the fact that hyperbolic groups quasi–isometrically embed into ℓ^1 (and hence coarsely into ℓ^2). The message is that by using a strongly hyperbolic metric, the embedding into $\ell^1\Gamma$ is very explicit and straightforward.

1. INTRODUCTION

The fact that hyperbolic groups [6] coarsely embed into a Hilbert space was first proved by Sela [11]. Since then, many other proofs have emerged, usually proving a stronger property (Yu's Property A or finite asymptotic dimension [10]) and subsequently applying a general theorem that these imply the existence of coarse embeddings. This approach has some obvious advantages; however a notable disadvantage is that the resulting concrete embedding is usually not explicit and difficult to write down [7, 8, 4, 5].

On the other hand, for trees and genuine hyperbolic spaces it is easy to write down explicit quasi-isometric embeddings into ℓ^1 or L^1: the idea being to fix a point o in the space in question and then map a point x to the characteristic function of the interval $[ox]$, the latter notion suitably interpreted. (Let us remind the reader that ℓ^1 coarsely embeds into ℓ^2, although the composition with a quasi–isometric embedding into ℓ^1 rarely has the best possible compression.)

The point of this note is to present a very easy-to-write-down quasi–isometric embeddings of hyperbolic groups Γ into $\ell^1\Gamma$; just like in the case of a tree, the functions are "smoothed-out" characteristic functions of intervals. The traditional stumbling block of this approach (in some sense the existence of "ladders" of geodesics in hyperbolic groups) is circumvented by assuming that we have a roughly geodesic, strongly hyperbolic metric on Γ [9]. Such always exist in the quasi–isometry class of a word metric; for instance the Green metric [1, 2, 9], which is particularly easy to construct.

We opted to include all the details of the norm estimates, so as to make sure that we do not use results relying on the assumption of existence of geodesics; and also to make it easy to see how do the roughly geodesic, and strongly hyperbolic assumptions come in.

Finally, let us note that B. Nica has constructed such straightforward embeddings using the boundary (private communication).

2. PRELIMINARIES

When we talk about metric spaces, and there is no possible confusion about the metric, we shall denote the distance between two points z, w by $|zw|$ or $|z, w|$.

2.1. **Quasi–isometric embeddings.** A (not necessarily continuous) function $F : \Gamma \to \Omega$ between metric spaces Γ and Ω is called a *quasi–isometric embedding*, if there exist constants $K > 0$, $L \geq 0$, such that $\frac{1}{K}|z,w| - L \leq |F(z), F(w)| \leq K|z,w| + L$ for all $z, w \in \Gamma$.

2.2. **Roughly geodesic spaces.** Given a metric space Γ, a *rough geodesic* (with parameter $C \geq 0$) between $x, y \in \Gamma$ is a (not necessarily continuous) map $\gamma : [u,v] \to \Gamma$, such that $\gamma(u) = x$, $\gamma(v) = y$, and $|s - s'| - C \leq |\gamma(s), \gamma(s')| \leq |s - s'| + C$ for all $s, s' \in [u, v]$.

A metric space Γ is said to be *roughly geodesic*, if there exists $C \geq 0$, such that there exists a rough geodesic with parameter C between any two points of Γ.

This is a notion between geodesics (when $C = 0$) and quasi–geodesics (quasi–isometrically embedded intervals). The hyperbolic theory works virtually without any changes when the classic geodesic assumption is relaxed to roughly geodesic [3].

2.3. **Gromov product and double difference.** These are matters of notation. For a three points a, b, o in a metric space, the *Gromov product* of a and b, with basepoint o, is the number

$$\langle a, b\rangle_o = \tfrac{1}{2}\left(|a,o| + |b,o| - |a,b|\right).$$

For a four points a, a', b, b' in a metric space, we define their *double difference* to be the expression

$$\langle a, a'|b, b'\rangle = \tfrac{1}{2}\left(|a,b| - |a',b| - |a,b'| + |a',b'|\right).$$

Among the identities about this quantity that can be proved just by expanding the sides, are these:

$$\langle a, a'|b, b'\rangle = \langle b, b'|a, a'\rangle = -\langle a, a'|b', b\rangle; \tag{1}$$

$$\langle a, a'|b, b'\rangle = \langle a, b'\rangle_b - \langle a', b'\rangle_b. \tag{2}$$

2.4. **Strong hyperbolicity.** Recall that a metric space Γ is said to be *δ-hyperbolic*, if the following inequality holds for any four points $x, y, z, w \in \Gamma$:

$$\langle x, z\rangle_w \geq \min\left(\langle x, y\rangle_w, \langle y, z\rangle_w\right) - \delta.$$

This condition is equivalent to requiring that for any four points $x, y, z, w \in \Gamma$, the following inequality holds:

$$|xy| + |zw| \leq \max\left(|xz| + |yw|, |yz| + |xw|\right) + 2\delta. \tag{3}$$

In the paper [9], the authors define a more refined version of hyperbolicity, called *strong hyperbolicity*. Instead of the definition, we recall an equivalent formulation [9, Lemma 6.2].

A δ-hyperbolic metric space Γ is *strongly hyperbolic*, if there exist $L, \lambda > 0$ and $R_0 > 0$, such that for any $x, y, w, o \in \Gamma$,

$$\langle x, o|w, y\rangle \geq R \geq R_0 \quad \Longrightarrow \quad |\langle x, y|w, o\rangle| \leq L\exp(-\lambda R).$$

Note that the conclusion estimate is symmetric in x and y, hence the same conclusion is true if we assume $\langle y, o|w, x\rangle \geq R \geq R_0$ instead.

3. The map

Let Γ be a metric space and fix a point $o \in \Gamma$. For each $x \in \Gamma$, define a function $\xi_x : \Gamma \to [0, \infty)$ by

$$\xi_x(w) = \exp\left(-A\langle x, o\rangle_w\right), \quad w \in \Gamma,$$

where A is a constant we declare more precisely later on.

One possible way to think about the ξ_x is that it is roughly a "smoothed–out characteristic function of the interval $[o, x]$". The point is that $\langle x, o\rangle_w$ is close to 0 for points between o and x; while it decreases when w moves away.

Proposition 3.1. *Let Γ be a countable, roughly geodesic metric space, endowed with a strongly hyperbolic metric. Assume that it has at most exponential growth. Then there exists $A > 0$, such that the map $\Xi : \Gamma \to \ell^1\Gamma$, $\Xi(x) = \xi_x$, is a quasi-isometric embedding.*

Let us recall that Γ has at most *exponential growth*, if there exists a constant $G_1 \geq 0$, such that $|B(w, t)| \leq \exp(G_1 t)$ for any $t \geq 0$ and $w \in \Gamma$.

For most of the properties of Ξ asserted above assuming the standard hyperbolicity is sufficient; however strong hyperbolicity is used in showing that $\|\xi_x - \xi_y\|_1$ is bounded from above linearly in $|x, y|$.

4. Proof

The remaining part of the paper contains a computation proving Proposition 3.1. For completeness, all details are provided. However, as usual with hyperbolic matters, the "meat" of the argument is relatively easily explainable with a few drawings. Hence, the reader is highly encouraged to scribble a few tripods and their four–leaved analogues while reading the claims.

We are assuming throughout that Γ is δ-hyperbolic and roughly geodesic; however we only add the "strongly" assumption when explicitly needed.

4.1. **Notation.** Given $x, y \in \Gamma$ and $\alpha \geq 0$, denote $N(x, y; \alpha) = \{w \in \Gamma \mid \langle x, y\rangle_w \leq \alpha\}$. These could be thought of as "fattened intervals" between x and y.

4.2. **Preparation.** Running the danger of reproducing the arguments already in the literature, we include here proofs of some well–known properties of hyperbolic spaces. However we do not assume existence of geodesics, and the only way rough geodesics are used in this subsection are through the following Lemma: an analogue of the "minsize" definition of hyperbolicity.

Lemma 4.1. *There exists $\alpha_0 \geq 0$, such that for any $\alpha \geq \alpha_0$, and any $x, y, z \in \Gamma$,*

$$N(x, y; \alpha) \cap N(y, z; \alpha) \cap N(z, x; \alpha) \neq \emptyset.$$

The fact we need next is that the "intervals" $N(x, y; \alpha)$ are not fat; in some sense an analogue of the Morse lemma.

Lemma 4.2. *Let $o, q \in \Gamma$, $p_1, p_2 \in N(o, q; \alpha)$. Then $|p_1p_2| \leq 2\delta + 2\alpha + ||qp_1| - |qp_2||$.*

Proof. By the choice of p_i, we have

$$|op_i| = 2\langle o, q\rangle_{p_i} + |oq| - |qp_i| \leq 2\alpha + |oq| - |qp_i|.$$

Thus

$$|qp_2| + |op_1| \leq |oq| + 2\alpha + ||qp_1| - |qp_2|| \,, \quad \text{and}$$
$$|qp_1| + |op_2| \leq |oq| + 2\alpha + ||qp_1| - |qp_2|| \,.$$

Thus, using the hyperbolicity inequality (3),

$$\begin{aligned} |p_1p_2| + |oq| &\leq 2\delta + \max\left(|qp_2| + |op_1|, |qp_1| + |op_2|\right) \\ &\leq 2\delta + 2\alpha + |oq| + ||qp_1| - |qp_2|| \,. \end{aligned}$$

The required inequality follows. □

The next Lemma resembles the statement that the Gromov product $\langle o, q\rangle_w$ roughly expresses the distance from w to a geodesic between o and q.

Lemma 4.3. *Let $o, q, w \in \Gamma$, and let $p \in N(o, q; \alpha) \cap N(q, w; \alpha) \cap N(w, o; \alpha)$. Then $|pw| - 2\alpha \leq \langle o, q\rangle_w \leq |pw| + \alpha$.*

Proof. First, using the triangle inequality and the choice of p, we have

$$\begin{aligned} 2\langle o, q\rangle_w &= |ow| + |qw| - |oq| \\ &\leq (|op| + |pw|) + (|qp| + |pw|) + (2\langle o, q\rangle_p - |op| - |qp|) \\ &\leq 2|pw| + 2\alpha. \end{aligned}$$

This proves the right-hand inequality. For the left-hand one, using again the choice of p and the triangle inequality, we estimate

$$\begin{aligned} 2|pw| &\leq (2\langle q, w\rangle_p - |qp| + |qw|) + (2\langle o, w\rangle_p - |op| + |ow|) \\ &\leq 4\alpha - (|qp| + |op|) + |qw| + |ow| \\ &\leq 4\alpha - |oq| + |qw| + |ow| \\ &= 4\alpha + 2\langle o, q\rangle_w. \end{aligned}$$

This finishes the proof. □

Lemma 4.4. *Given $o, q \in \Gamma$ and $\alpha \geq \alpha_0$ (so that Lemma 4.1 applies), there exists a set $P \subseteq N(o, q; \alpha)$, such that $|P| \leq \frac{1}{\alpha}|oq| + 2$ and for every $w \in \Gamma$ there exists $p \in P$, such that $\langle o, q\rangle_w \geq |pw| - 5\alpha - 2\delta$.*

Proof. By Lemmas 4.1 and 4.3, for any $w \in \Gamma$ there exists a $p_w \in N(o, q; \alpha)$, such that $\langle o, q\rangle_w \geq |p_w w| - 2\alpha$. Collect all those p_w, $w \in \Gamma$, into a subset of $N(o, q; \alpha)$ and let P be a maximal $(2\delta + 3\alpha)$-separated subset of it. If $p_1 \neq p_2 \in P$, Lemma 4.2 implies that $||qp_1| - |qp_2|| \geq \alpha$. Furthermore, as $\langle o, q\rangle_p \leq \alpha$, we have $|qp| \leq |oq| + \alpha$ for all $p \in P$. Consequently, we must have $\alpha(|P| - 1) \leq |oq| + \alpha$, hence $|P| \leq \frac{1}{\alpha}|oq| + 2$.

For the second part, note that by maximality of P, for any $w \in \Gamma$ there exists a $p \in P$ with $|pp_w| \leq 2\delta + 3\alpha$. Thus $\langle o, q\rangle_w \geq |p_w w| - 2\alpha \geq (|pw| - 3\alpha - 2\delta) - 2\alpha$. □

Recall that we are assuming at most exponential growth, i.e. there exists a constant $G_1 \geq 0$, such that $|B(w, t)| \leq \exp(G_1 t)$ for any $t \geq 0$ and $w \in \Gamma$.

Lemma 4.5. *There exists $G_2, G_3 \geq 0$, such that $|N(x, y; \alpha)| \leq G_3(|x, y| + 1)\exp(G_2\alpha)$ for all $\alpha \geq 0$ and $x, y \in \Gamma$.*

Proof. The argument is similar to the previous proof: choose a maximal $(2\delta+2\alpha+1)$-separated subset P of $N(x,y;\alpha)$. Using Lemma 4.2 we obtain $|P| \leq |xy| + \alpha + 1$, and hence

$$|N(x,y;\alpha)| \leq |P| \max_{p\in P} (|B(p, 2\delta+2\alpha+1)|) \leq (|xy| + \alpha + 1)\exp(G_1(2\delta+2\alpha+1)),$$

which clearly admits a bound of the desired sort. □

4.3. **Well-definedness, i.e.** $\xi_x \in \ell^1\Gamma$**.**

Lemma 4.6. *For every* $x \in \Gamma$*,* $\|\xi_x\|_1 \leq \frac{G_3}{A-G_2}(|x,o|+1)$*, provided* $A > G_2$*.*

Proof. Summing by the level sets of $\langle x, o\rangle_w$, and using Lemma 4.5, we get

$$\begin{aligned}\|\xi\|_1 &\leq \int_0^\infty \exp(-At)|N(x,0;t)|dt \\ &\leq G_3(|x,o|+1)\int_0^\infty \exp((G_2-A)t)dt \\ &\leq \frac{G_3}{A-G_2}(|x,o|+1).\end{aligned}$$

We are done. □

4.4. **A lower bound on** $\|\xi_x - \xi_y\|_1$**.** For the rest of the paper, we fix $x, y \in \Gamma$.

The proof follows the natural idea, namely that the points in an "interval" between x and y contribute to the norm $\xi_x - \xi_y$ substantially, whence the norm is proportional to $|x,y|$. Loosely speaking, for any point w in an "interval" between x and o, the Gromov product $\langle x, o\rangle_w$ is close to 0, thus the value $\xi_x(w)$ is close to 1. Also, the values $\xi_x(w)$ decrease exponentially as w moves away from the "interval" between x and o. Finally, the assumption of rough geodesicity assures that there are sufficiently many points in any of the "intervals". At this point, one draws the usual picture of a "triangle" oxy, close to a tripod, and the outline of a proof should become clear.

Here is a proof. Recall that we can think of $N(x,y;\alpha)$ as of a (fattened) interval between x and y; we shall consider a triangle oxy with these sets as "sides". The next Lemma says, in this language, that triangles are slim; the following Lemma will say that points along the parts sticking out towards x and y contribute a fixed amount towards the norm $\|\xi_x - \xi_y\|_1$.

Lemma 4.7. *For any* $\alpha \geq 0$*,* $N(x,y;\alpha) \setminus N(o,x;\alpha+\delta) \subseteq N(o,y;\alpha+\delta)$*.*

Proof. This is seen by using the definition of hyperbolicity:

$$\alpha + \delta \geq \langle x, y\rangle_w + \delta \geq \min(\langle x, o\rangle_w, \langle y, o\rangle_w).$$

But as $\langle x, o\rangle_w > \alpha + \delta$, it must be the case that $\langle y, o\rangle_w \leq \alpha + \delta$. □

Lemma 4.8. *Let* $\alpha \geq 0$*. Then there exists a constant* $T > 0$*, such that for any* $w \in N(x,y;\alpha) \setminus N(o,x;\alpha+2\delta)$*, we have* $|\xi_x(w) - \xi_y(w)| \geq T$*.*

Proof. For such a w, we have that $\langle o, y\rangle_w \leq \alpha+\delta$ by Lemma 4.7, and of course $\langle o, x\rangle_w > \alpha + 2\delta$. Hence

$$\begin{aligned}|\xi_x(w) - \xi_y(w)| &\geq \exp(-A\langle o, y\rangle_w) - \exp(-A\langle o, x\rangle_w) \\ &\geq \exp(-A(\alpha+\delta)) - \exp(-A(\alpha+2\delta)).\end{aligned}$$

The last expression is a positive number, we can call it T. □

We now need to prove that we have sufficiently many contributing points w as in the previous Lemma. The first preparatory Lemma roughly says that a point y is about $\langle o, x\rangle_y$–far from an interval between o and x.

Lemma 4.9. *For any $\alpha \geq 0$, if $z \in N(x, o; \alpha)$, then $|y, z| \geq \langle o, x\rangle_y - \alpha$.*

Proof. This follows from a general inequality for Gromov products:

$$|\langle o, x\rangle_y - \langle o, x\rangle_z| \leq |y, z|.$$

□

Given a constant C of roughness of geodesics, there exists $\alpha_1 \geq 0$, such that any such rough geodesic between any pair of points $a, b \in \Gamma$ is contained in $N(a, b; \alpha_1)$. Furthermore, any such rough geodesic can "jump" only over distance C, so that there is at least $\frac{1}{C}|a, b|$ points on any rough geodesic between $a, b \in \Gamma$. All this follows directly from the definition of rough geodesics.

We are assuming that Γ is roughly geodesic, so that between any pair of points $a, b \in \Gamma$, there exists a rough geodesic (with constant C).

From now on, fix an $\alpha \geq \alpha_1$. There exists a rough geodesic between x and y, and it is contained in $N(x, y; \alpha)$. By Lemma 4.9, there are at least $\frac{1}{C}(\langle o, x\rangle_y - \alpha - 2\delta)$ points on it within $N(x, y; \alpha) \setminus N(o, x; \alpha+2\delta) \subseteq N(o, y; \alpha+\delta)$ (see Lemma 4.7). Each of those contributes at least T towards $\|\xi_x - \xi_y\|_1$ by Lemma 4.8. Analogously (with x and y swapped), there are at least $\frac{1}{C}(\langle o, y\rangle_x - \alpha - 2\delta)$ more points on it, different from the ones found before, because these are in $N(x, y; \alpha) \setminus N(o, y; \alpha + 2\delta) \subseteq N(o, x; \alpha + \delta)$. Thus altogether

$$\|\xi_x - \xi_y\|_1 \geq \tfrac{T}{C}\left(\langle o, x\rangle_y + \langle o, y\rangle_x - 2\alpha - 4\delta\right) = \tfrac{T}{C}\left(|x, y| - 2\alpha - 4\delta\right).$$

We have the desired lower bound.

4.5. An upper bound on $\|\xi_x - \xi_y\|_1$.

Let us recall that $x, y, o \in \Gamma$ are fixed.

We shall split Γ into two subsets, and estimate $|\xi_x(w) - \xi_y(w)|$ on these separately. The sets can be roughly understood geometrically as follows: if $w \in \Gamma$, consider a/the projection of w onto a "tripod" oxy. The first set contains those w for which this projection lands reasonably close to the "side" between x and y; leaving those that project onto the "arm" towards o for the second set.

To define the first set, let $R \geq 0$ and let

$$A_R = \{w \in \Gamma \mid \langle x, o|w, y\rangle \leq R \text{ and } \langle y, o|w, x\rangle \leq R\}.$$

Lemma 4.10. *Let $R \geq 0$. Then*

$$\|\xi_x|_{A_R} - \xi_y|_{A_R}\|_1 \leq 2e^{AR}\tfrac{G_3}{A-G_2}(|x, y| + 1),$$

provided $A > G_2$.

Proof. Using (2) we get, for $w \in A_R$, that

$$\begin{aligned}\langle x, y\rangle_w &= \langle y, o\rangle_w + \langle x, o|t, y\rangle \leq \langle y, o\rangle_w + R, \text{ and similarly}\\ \langle x, y\rangle_w &\leq \langle x, o\rangle_w + R.\end{aligned}$$

Consequently

$$\begin{aligned}|\xi_x(w) - \xi_y(w)| &\leq \exp(-A\langle x, o\rangle_w) + \exp(-A\langle y, o\rangle_w)\\ &\leq 2e^{AR}\exp(-A\langle x, y\rangle_w).\end{aligned}$$

Now performing the same estimate as in the proof of Lemma 4.6 for the function $\zeta : \Gamma \to [0, \infty)$ given by $\zeta(w) := \exp(-A\langle x, y\rangle_w)$, we get

$$\|\xi_x|_{A_R} - \xi_y|_{A_R}\|_1 \leq 2e^{AR}\|\zeta\|_1 \leq 2e^{AR}\tfrac{G_3}{A-G_2}(|x, y| + 1),$$

provided $A > G_2$. □

The goal for the rest of the subsection is to prove that for a sufficiently large R, the norm $\left\|\xi_x|_{\Gamma\setminus A_R} - \xi_y|_{\Gamma\setminus A_R}\right\|_1$ is bounded by a constant; this (together with Lemma 4.10) will provide a linear upper bound on $\|\xi_x - \xi_y\|_1$.

The strategy is to convert the quantities involved into $\langle o, w\rangle_q$.

Lemma 4.11. *If $q \in N(x, y; \alpha) \cap N(o, x; \alpha) \cap N(o, y; \alpha)$, then*

$$\langle x, o|w, y\rangle \leq \langle o, w\rangle_q + \alpha \tag{4}$$

for every $w \in \Gamma$. Furthermore, if in addition $\langle x, o|w, y\rangle > 2\alpha + \delta$, then also

$$\langle o, w\rangle_q - 2\alpha - \delta \leq \langle x, o|w, y\rangle. \tag{5}$$

Proof. We shall proceed in several steps.

Step 1. $\langle x, o|w, y\rangle \leq \langle x, o|w, q\rangle + \alpha$.

Expanding and using the triangle inequality, we get

$$\begin{aligned}2\langle x, o|w, y\rangle &= |xw| + |oy| - |xy| - |ow|\\ &\leq |xw| + (|oq| + |yq|) + (2\langle x, y\rangle_q - |xq| - |yq|) - |ow|\\ &\leq 2\alpha + 2\langle x, o|w, q\rangle.\end{aligned}$$

Step 2. $\langle x, o|w, q\rangle \leq \langle o, w\rangle_q$ (together with the previous Step, this proves one of the claims).

Using (1) and (2) we have

$$\langle x, o|w, q\rangle = \langle o, x|q, w\rangle = \langle o, w\rangle_q - \langle x, w\rangle_q \leq \langle o, w\rangle_q.$$

Step 3. $\langle x, w\rangle_q \leq \alpha + \delta$.

Writing out the hyperbolic inequality, we have

$$\alpha + \delta \geq \langle x, o\rangle_q + \delta \geq \min\left(\langle x, w\rangle_q, \langle o, w\rangle_q\right).$$

But by the assumption and the already proved (4), we have $\langle o, w\rangle_q \geq \langle x, o|w, y\rangle - \alpha > \alpha + \delta$. Thus the minimum in the above displayed inequality must be achieved at $\langle x, w\rangle_q$ and we are done.

Step 4. $\langle x, o|w, y\rangle \geq \langle o, w\rangle_q - 2\alpha - \delta$.

Applying the previous step, the assumption, and simply expanding to obtain the first equality, we get

$$\begin{aligned}\langle x, o|w, y\rangle &= -\langle x, w\rangle_q - \langle o, y\rangle_q + \langle x, y\rangle_q + \langle o, w\rangle_q \\ &\geq -(\alpha+\delta) - \alpha + \langle o, w\rangle_q.\end{aligned}$$

We are done. □

Remark 4.12. It follows from the previous proof that if $\langle x, o|w, y\rangle = R > 2\alpha + \delta$, then also $\langle y, o|w, x\rangle$ satisfies the same inequality (5) and thus $\langle y, o|w, x\rangle \geq \langle x, o|w, y\rangle - 3\alpha - \delta$. The reasoning is as follows: from (4) we have $\langle o, w\rangle_q \geq R - \alpha$. The proof of Step 3 applied to y instead of x now yields $\langle y, w\rangle_q \leq \alpha + \delta$. A proof analogous to Step 4 now gives $\langle y, o|w, x\rangle \geq \langle o, w\rangle_q - 2\alpha - \delta \geq R - 3\alpha - \delta$.

An inequality like (4) holds for $\langle y, o|w, x\rangle$ automatically just by re-labelling.

Lemma 4.13. *Let $q \in N(x, y; \alpha) \cap N(o, x; \alpha) \cap N(o, y; \alpha)$ and $\langle x, o|w, y\rangle \geq 2\alpha + \delta$. Then $\langle x, o\rangle_w \geq \langle q, o\rangle_w - \alpha - \delta$.*

Proof. As the assumptions of Lemma 4.11 are satisfied, we can use Step 3 from its proof, i.e. that $\langle x, w\rangle_q \leq \alpha + \delta$. Hence

$$\begin{aligned}2\langle x, o\rangle_w &= |wx| + |wo| - |xo| \\ &= (-2\langle x, w\rangle_q + |xq| + |wq|) + |wo| - |xo| \\ &\geq -2\alpha - 2\delta + |xq| + |wq| + |wo| - (|xq| + |qo|) \\ &= -2\alpha - 2\delta + 2\langle q, o\rangle_w.\end{aligned}$$

□

Lemma 4.14. *Let $w_1, w_2 \in \Gamma$ and let $p_i \in N(o, q; \alpha) \cap N(o, w_i; \alpha) \cap N(q, w_i; \alpha)$. Then*

$$|p_1 p_2| \leq |\langle o, w_1\rangle_q - \langle o, w_2\rangle_q| + 8\alpha + 2\delta.$$

Proof. Using Lemma 4.2 and Lemma 4.3, we estimate

$$\begin{aligned}|p_1 p_2| &\leq 2\delta + 2\alpha + ||qp_1| - |qp_2|| \\ &\leq 2\delta + 2\alpha + 6\alpha + |\langle o, w_1\rangle_q - \langle o, w_2\rangle_q|.\end{aligned}$$

□

We shall split $\Gamma \setminus A_R$ further into the following sets:

$$B_n = \{w \in \Gamma \mid n \leq \max(\langle x, o|w, y\rangle, \langle y, o|w, x\rangle) < n+1\},$$

for $n \in \mathbb{N}$ with $n \geq R - 1$ (strictly speaking there may be some overlap with A_R, but this does not affect the norm estimate).

Fix, from now on, an $\alpha \geq \alpha_0$ (see Lemma 4.1) and $R \geq R_0$ (from the definition of strong hyperbolicity) and $R \geq 5\alpha + 2\delta$.

Lemma 4.15. *Given $n \in \mathbb{N}$ with $n \geq R - 1$, there exists $p_n \in N(o, q; \alpha)$, such that for any $w \in B_n$, $|p_n w| \leq \langle x, o\rangle_w + 20\alpha + 6\delta$, and similarly for $\langle y, o\rangle_w$.*

Proof. By our choice of α, for any $w \in B_n$ there is a $p_w \in N(o,q;\alpha) \cap N(o,w;\alpha) \cap N(q,w;\alpha)$ by Lemma 4.1. By the choice of R and Remark 4.12, $\langle x,o|w,y\rangle$ and $\langle y,o|w,x\rangle$ are within $6\alpha + 2\delta$ of each other. Hence by Lemma 4.13 and Lemma 4.3, we have $|p_w w| \leq 2\alpha + (\alpha+\delta) + \langle x,o\rangle_w$, and likewise with $\langle y,o\rangle_w$.

To find a common p_n that works for all $w \in B_n$, Lemma 4.11 and Lemma 4.14 imply that for $w_1, w_2 \in B_n$, we have $|p_{w_1} p_{w_2}| \leq (8\alpha+2\delta) + (3\alpha+\delta) + (6\alpha+2\delta) + 1$. Thus the set $\{p_w \in N(o,q;\alpha) \mid w \in B_n\}$ has diameter at most $17\alpha + 5\delta$. So we can choose p_n to be arbitrary one of the p_w, and we shall have $|p_n w| \leq \langle x,o\rangle_w + 20\alpha + 6\delta$ (similarly for $\langle y,o\rangle_w$) for any $w \in B_n$. □

Coming back to the norm estimate we are after, on one hand we will use a simple bound using Lemma 4.15:

$$\begin{aligned} |\xi_x(w) - \xi_y(w)| &= |\exp(-A\langle x,o\rangle_w) - \exp(-A\langle y,o\rangle_w) \\ &\leq 2\exp\left(A\cdot(20\alpha+6\delta)\right)\exp(-A|p_n w|) \end{aligned} \tag{6}$$

for any $w \in B_n$. On the other hand, basic calculus tells us that for any $w \in \Gamma$,

$$\begin{aligned} |\xi_x(w) - \xi_y(w)| &= |\exp(-A\langle x,o\rangle_w) - \exp(-A\langle y,o\rangle_w) \\ &\leq A|\langle x,o\rangle_w - \langle y,o\rangle_w| \\ &= |\langle x,y|w,o\rangle|. \end{aligned} \tag{7}$$

By the choice of R, we can apply the estimate from the definition of strong hyperbolicity to continue (7) to get

$$|\xi_x(w) - \xi_y(w)| \leq L\exp(-\lambda n)$$

for any $w \in B_n$. Putting the inequalities together, and using the assumption of exponential growth, we get

$$\begin{aligned} \|\xi_x|_{B_n} - \xi_y|_{B_n}\|_1 &= \sum_{w\in B_n} |\xi_x(w) - \xi_y(w)| \\ &= \sum_{w\in B_n : |p_n w| < \frac{\lambda n}{2G_2}} |\xi_x(w) - \xi_y(w)| + \sum_{w\in B_n : |p_n w| \geq \frac{\lambda n}{2G_2}} |\xi_x(w) - \xi_y(w)| \\ &\leq L\exp(-\lambda n)\exp\left(G_2\cdot\tfrac{\lambda n}{2G_2}\right) + \int_{\frac{\lambda n}{2G_2}}^{\infty} M\exp((G_2 - A)t)\,dt \\ &= L\exp(-\tfrac{1}{2}\lambda n) + \tfrac{M}{A-G_2}\exp\left(\tfrac{(G_2-A)\lambda n}{2G_2}\right), \end{aligned}$$

where $M = 2\exp\left(A\cdot(20\alpha+6\delta)\right)$. The upshot is that the last expression is summable over $n \geq R-1$ (when $A > G_2$), hence

$$\left\|\xi_x|_{\Gamma\setminus A_R} - \xi_y|_{\Gamma\setminus A_R}\right\|_1 \leq \sum_{n\geq R-1} \|\xi_x|_{B_n} - \xi_y|_{B_n}\|_1$$

is bounded from above by a constant. This finishes the proof.

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