METRIC SPARSIFICATION PROPERTY AND LIMIT OPERATORS

ABSTRACT. We extend the result of Lindner and Seidel [3] for $\mathbb{Z}^n$ to all exact discrete groups. In other words, we show that if Γ is an exact discrete group and T is a band–dominated operator on $\ell^p(\Gamma)$, such that all the limit operators of T are invertible, then the norms of their inverses are automatically uniformly bounded.

1. INTRODUCTION

The limit operator theory (initiated in [5], see also the book [6] and the paper [3] for a recent list of relevant references) studies Fredholmness for the class of *band–dominated operators* on various ℓ^p-spaces over $\mathbb{Z}^n$ in terms of the so-called *operator spectrum*, i.e. the collection of *limit operators* associated with an operator in question.

The setup is roughly as follows: Thinking of an operator T in $\ell^p(\mathbb{Z}^n)$ as a $\mathbb{Z}^n$-by-$\mathbb{Z}^n$ matrix, we say that T is a *band operator* if the only non-zero entries in its matrix appear within a fixed distance from the diagonal. The *band–dominated* operators in $\ell^p(\mathbb{Z}^n)$ are then norm–limits of band operators. Given a band–dominated operator T and a sequence $(g_n)_{n\in\mathbb{N}} \subset \mathbb{Z}^n$ converging to infinity, the sequence of *shifts* of T by g_n always contains a strongly convergent subsequence, and the strong limit is called a *limit operator* of T, associated with (g_n). The collection of all limit operators of T is called the *operator spectrum*, denoted $\sigma_{\mathrm{op}}(T)$. The basic theorem in this theory is the following:

Theorem 1.1. *Let T be a band–dominated operator in $\ell^p(\mathbb{Z}^n)$. Then T is Fredholm if and only if all $S \in \sigma_{\mathrm{op}}(T)$ are invertible and their inverses are uniformly bounded in norm.*

The main result of the paper [3] is that one can remove the requirement of the uniform boundedness of inverses in the above theorem.

John Roe [7] has explained the connection between the above setup and coarse geometry in the Hilbert space case (i.e. $p = 2$): Coarse geometers call the band operators as *finite propagation operators* and the collection of all band-dominated operators comprises the *translation C^*-algebra* of $\mathbb{Z}^n$ (also called the *uniform Roe algebra* of $\mathbb{Z}^n$ in the literature). John Roe extended the symbol calculus implicit in Theorem 1.1 to all discrete groups Γ and proved the Fredholmness criterion 1.1 for all *exact* Γ's. Furthermore, the results of Roe and Willett [8] show that this approach is *not* going to provide a Fredholmness criterion for non-exact groups Γ.

Summarising, John Roe has established that the limit operator theory setup is inherently *coarse geometric* in nature, and that one may expect that the operator theoretic phenomena of band–dominated operators come from large–scale geometry of the underlying discrete group.

In this note, we establish one more example of this philosophy: we extend the result of [3] to all exact discrete groups. More precisely, we show that the "main tool" of Lindner an Seidel,

Date: April 7, 2014.

namely [3, Proposition 6], holds for all exact discrete groups, not just $\mathbb{Z}^n$. The second part of their proof carries over without significant changes.

2. Definitions and Notation: Coarse Geometry

2.1. Coarse Geometry. Let X be a metric space. We shall denote the balls in X as

$$B_x(R) = \{y \in X \mid d(x,y) \leq R\} \qquad \text{for } x \in X \text{ and } R \geq 0$$

and for a set $U \in X$ and $m \geq 0$, we will denote

$$N_m(U) = \{x \in X \mid d(x,U) \leq m\}$$

the m-neighbourhood of U.

We say that it is *uniformly discrete* if there exists $c > 0$, such that $x \neq y$ implies $d(x,y) \geq c$ for $x,y \in X$. We say that X has bounded geometry, if for every $R \geq 0$, $\sup_{x \in X} |B_x(R)| < \infty$.

The examples of uniformly discrete metric spaces with bounded geometry that we shall mainly consider in this note are countable discrete groups. Any such group Γ can be endowed with a proper left-invariant metric (unique up to coarse equivalence), turning it into a uniformly discrete metric space with bounded geometry. If Γ is finitely generated, then a choice of a generating set provides us with a concrete *word metric* [4]. For example, if $\Gamma = \mathbb{Z}^n$, choosing the generating set consisting of the "natural basis" $\{(1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}$ yields the usual "absolute value" metric: $d((k_1,\ldots,k_n),(l_1,\ldots,l_n)) = |l_1 - k_1| + \cdots + |l_n - k_n|$.

The coarse geometric definitions make sense for general metric spaces X; however for the purposes of this note the reader should keep in mind the main example, namely discrete groups.

The definition of exactness of discrete groups that we shall use in this note goes under the name of the Metric Sparsification Property (see [1] and [10] for the equivalences of definitions). It was introduced in [2] precisely for the purposes of "localising estimating the operator norm". The gist of the property is that one can choose "big sets" (in a given measure) that split into well separated uniformly bounded sets.

Definition 2.1. Let (X,d) be a metric space. Then X has *metric sparsification property* (MSP) with constant $c \in (0,1]$, if there exists a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$, such that for all $m \in \mathbb{N}$ and any finite positive Borel measure μ on X, there is a Borel subset $\Omega = \sqcup_{i \in I} \Omega_i$ of X, such that

- $d(\Omega_i, \Omega_j) \geq m$ whenever $i \neq j \in I$;
- $\mathrm{diam}(\Omega_i) \leq f(m)$ for every $i \in I$;
- $\mu(\Omega) \geq c\,\mu(X)$.

3. Definitions and Notation: Operator Theory

We shall mostly follow the notation and terminology from [3].

3.1. Spaces and Operators. Let X be a uniformly discrete metric space with bounded geometry and let E be a Banach space. We shall use the notation $\ell^p_E(X) = \ell^p(X,E)$ for the Banach space of p-summable functions on X with values in E, $p \in \{0\} \cup [1,\infty]$ (endowed with the natural p-norm, denoted by $||\cdot||_p$). The traditional space considered in coarse geometry is $\ell^2_H(X)$, with H being a separable Hilbert space, which is naturally also a Hilbert space. Note that we think of the vectors $\psi \in \ell^p_E(X)$ as functions on X, thus we can talk about their support $\mathrm{supp}(\psi) = \{x \in X \mid \psi(x) \neq 0\}$.

We shall denote by $\mathcal{L}(K)$ the algebra of bounded linear operators on a Banach space K. We will think of the operators in $\mathcal{L}(\ell^p_E(X))$ as X-by-X matrices with entries in $\mathcal{L}(E)$.

For a subset $Y \subset X$, we denote by $P_Y \in \mathcal{L}(\ell^p_E(X))$ the "projection onto $\ell^p_E(Y) \subset \ell^p_E(X)$", i.e. an operator represented by a diagonal matrix consisting of 1_E at (y,y)-entries for $y \in Y$ and 0 elsewhere. Note that with this notation, the (x,y)-entry of the matrix representing $T \in \mathcal{L}(\ell^p_E(X))$ can be expressed as $P_{\{x\}} T P_{\{y\}}$.

Definition 3.1. For $T \in \mathcal{L}(\ell^p_E(X))$, we say that T has *propagation* at most $R \geq 0$, if the (x,y)-entry of T is 0 whenever $d(x,y) > R$.

Operators with finite propagation are also called *band operators* (usually considered only the case $X = \mathbb{Z}^n$). Norm limits of these are called *band–dominated* operators. To keep the notation from [3], we denote by $\mathcal{A}_p$ the band–dominated operators on $\ell^p_E(X)$; it is a Banach algebra.

3.2. Lower norm.

Definition 3.2. Let K, L be Banach spaces and $T \in \mathcal{L}(K,L)$. We define the *lower norm* of T to be

$$\nu(T) = \inf\left\{ \frac{\|T\psi\|_L}{\|\psi\|_K} \mid \psi \in K \setminus \{0\} \right\}.$$

If $K = L = \ell^p_E(X)$ and $D \geq 0$, we shall also denote the lower norm computed on D-supported vectors by

$$\nu_D(T) = \inf\left\{ \frac{\|T\psi\|_p}{\|\psi\|_p} \mid \psi \in \ell^p_E(X) \setminus \{0\}, \mathrm{diam}(\mathrm{supp}(\psi)) \leq D \right\}.$$

Furthermore, if $F \subset X$ and $T \in \mathcal{L}(\ell^p_E(X))$, we shall denote the restriction of T to F by $T|_F = TP_F : P_F(\ell^p_E(X)) \cong \ell^p_E(F) \to \ell^p_E(X)$. The lower norms $\nu(T|_F)$ and $\nu_D(T|_F)$ shall be understood as the lower norms of $T|_F$ considered as an operator from $\ell^p_E(F)$ to $\ell^p_E(X)$.

3.3. Convergence. Apart from the operator norm on $\mathcal{L}(\ell^p_E(X))$, there is also a strong topology of sorts, which comes from considering operators in $\mathcal{L}(\ell^p_E(X))$ as X-by-X matrices. Roughly speaking, the topology would be the analogue of the strong topology on (infinite) matrices over X, but where the "entries" (operators on E) are considered with their natural norm topology. In [3] it is described as follows: let $X_1 \subset X_2 \subset \cdots$ be an exhaustion of X be finite sets, and denote the associated projections as $P_n = P_{X_n}$. We say that a sequence $(A_n) \subset \mathcal{L}(\ell^p_E(X))$ converges $\mathcal{P}$*-strongly* to an operator $A \in \mathcal{L}(\ell^p_E(X))$, if for every $m \in \mathbb{N}$:

$$\|P_m(A_n - A)\| + \|(A_n - A)P_m\| \to 0 \qquad \text{as } n \to \infty.$$

Similarly, the notion of compactness appropriate to this setup is "being approximable by finite matrices over X", but where we still allow the entries be arbitrary elements of $\mathcal{L}(E)$. More precisely, an operator $A \in \mathcal{L}(\ell^p_E(X))$ is $\mathcal{P}$*-compact*, if

$$\|(I - P_n)A\| + \|A(1 - P_n)\| \to 0 \qquad \text{as } n \to \infty.$$

Finally, note that the $\mathcal{P}$-notions do not depend on the choice of the exhaustion $X_1 \subset X_2 \subset \cdots$ of X.

3.4. **Limit operators and operator spectrum.** For the remainder of this section, we specialise to the case when $X = \Gamma$, a countable discrete group endowed with a proper, left-invariant metric. Each element $g \in \Gamma$ induces an isometry, "a shift by g", denoted by $U_g \in \mathscr{L}(\ell^p_E(\Gamma))$: it is defined by $U_g\psi(h) = \psi(g^{-1}h)$ for $\psi \in \ell^p_E(\Gamma)$ and $h \in \Gamma$. We say that a sequence $(g_n)_{n\in\mathbb{N}} \subset \Gamma$ converges to infinity, if $d(e, g_n) \to \infty$. Let us now fix an operator $A \in \mathscr{A}_p$. If $g \in \Gamma$, we can think of $U_{g^{-1}}AU_g$ as of *shifts* of A; in particular, this shifting does not increase the propagation. Now given any sequence (g_n) converging to infinity in Γ, consider the sequence of operators $(U_{g_n^{-1}}AU_{g_n})$. If it converges $\mathscr{P}$-strongly to an operator, say B, then we say that B is a *limit operator of A* associated to the sequence (g_n). The collection $\sigma_{\mathrm{op}}(A)$ of all limit operators of A is called the *operator spectrum* of A. Note that if $A \in \mathscr{A}_p$, then $\sigma_{\mathrm{op}}(A) \subset \mathscr{A}_p$.

Finally, A is said to be *rich*, if every sequence $(g_n) \subset \Gamma$ has a subsequence (g_{n_k}) such that A has a limit operator associated to (g_{n_k}). We denote by $\mathscr{A}^\$_p \subseteq \mathscr{A}_p$ the set of all rich, band–dominated operators. Note that given a rich operator A, $\sigma_{\mathrm{op}}(A)$ is $\mathscr{P}$-strongly sequentially compact (by a diagonal argument).

As an example, recall that the Roe C*-algebra $C^*(|\Gamma|, E)$ associated to (the Γ-module) $\ell^2_E(\Gamma)$ consists of norm–limits of finite propagation, locally compact operators. Recall that T is *locally compact* if all its entries are in $\mathscr{K}(E)$. An operator $A \in C^*(|\Gamma|, E)$ is rich if the collection of all its entries is compact as a subset of $\mathscr{K}(E)$. In other words, it is an element of the "standard" copy of the stabilisation of the uniform Roe algebra $C^*_u|\Gamma|\otimes\mathscr{K}(E)$ inside $C^*(|\Gamma|, E)$.

3.5. **Ghosts and symbol calculus.** For this subsection, we specialise to the case $p = 2$ and $E = \mathbb{C}$. John Roe, in [7], has explained that one can (and should) think of the operator spectrum $\sigma_{\mathrm{op}}(A)$ of a fixed operator $A \in \mathscr{A}_2$ as a $*$-strongly continuous function from the Stone-Čech boundary $\partial\Gamma$ to $\mathscr{A}_2$. Roughly speaking, a point $\omega \in \partial\Gamma$ represents a direction of "going off to infinity" in Γ, and to each such one associates the corresponding limit operator. This is captured in the "symbol calculus" sequence of C*-algebras

$$1 \to I \to \mathscr{A}_2 \xrightarrow{\sigma_{\mathrm{op}}} C_s(\partial\Gamma; \mathscr{A}_2) \to 1,$$

where $C_s(\partial\Gamma; \mathscr{A}_2)$ is the C*-algebra of $*$-strongly continuous functions from $\partial\Gamma$ to $\mathscr{A}_2$. This sequence is not quite exact: the symbol map σ_{op} is not surjective, as functions in its image are necesarily Γ-equivariant in the appropriate sense (see [7, Section 4]).

One of the points of [7] is that the ideal I consist precisely of *ghost operators*, which (applying [8]) agrees with the ideal of compact operators if and only if Γ is exact. Thus, for non-exact groups, a statement along the lines of Theorem 1.1, will not detect Fredholmness, but "invertibility modulo ghosts".

In fact, the construction from [8] provides us with a ghost operator $T \in \mathscr{A}_2$, which is positive, of norm one, and such that $T \geq P$, where P is a projection onto an infinite-dimensional subspace. Thus $\sigma_{\mathrm{op}}(1 - T)$ is the constant function 1 (i.e. every limit operator is the identity), but $1 - T$ has infinite-dimensional kernel.

4. Result

A straightforward adaptation of the argument that the Metric Sparsification Property implies the Operator Norm Localisation Property [2, Proposition 4.1] the following Proposition, a generalisation of [3, Proposition 6]. We remark that although the corresponding proof in [2] is formulated for $p=2$ and E a Hilbert space, it works just as well for other p's and Banach E's with the obvious modifications.

Proposition 4.1. *Let X be a uniformly discrete metric space with bounded geometry. Assume that X has the Metric Sparsification Property. For any $\delta>0$, $r\geq 0$ and $R\geq 0$ there exists $D\geq 0$, such that*

$$\nu(A|_F)\leq \nu_D(A|_F)\leq \nu(A|_F)+\delta$$

for any $A\in\mathcal{L}(\ell^p_E(X))$ with propagation at most R with $||A||_p\leq r$ and any $F\subset X$.

Proof. The first inequality is trivial. We focus on the second one, in the case when $F=X$ (for the sake of clarity). Fix $R,r\geq 0$ and an operator $A\in\mathcal{L}(\ell^p_E(X))$ with propagation at most R and norm at most r.

Step 1: Suppose that $\varphi\in\ell^p_E(X)\setminus\{0\}$ is such that its support splits into sufficiently separated subsets: $\varphi=\sum_{i\in I}\varphi_i$, $\varphi_i\neq 0$ for all $i\in I$ and $d(\mathrm{supp}(\varphi_i),\mathrm{supp}(\varphi_j))>2R$ if $i\neq j$. Then

$$\frac{||A\varphi||_p}{||\varphi||_p}\geq \inf_{i\in I}\frac{||A\varphi_i||_p}{||\varphi_i||_p}.$$

Indeed, since A can "spread the supports of vectors" only by at most R, the vectors $A\psi_i$ are still supported on mutually disjoint sets, hence they are mutually orthogonal[1]. Now suppose the inequality in the above display is false. Then

$$||A\varphi||_p^p=\sum_{i\in I}||A\varphi_i||_p^p>\sum_{i\in I}\frac{||A\varphi||_p^p||\varphi_i||_p^p}{||\varphi||_p^p}=\frac{||A\varphi||_p^p}{||\varphi||_p^p}\sum_{i\in I}||\varphi_i||_p^p=||A\varphi||_p^p,$$

which is a contradiction.

Step 2: Given a vector $\psi\in\ell^p_E(X)\setminus\{0\}$, we show that up to a uniformly estimated modification, we can split its support into well separated, uniformly bounded sets. Namely, let μ be the measure on X defined by declaring that the masses of points are $\mu(\{x\})=||\psi(x)||_E^p$. By the Metric Sparsification Property (with parameters c and f), there is a subset $\Omega=\sqcup_{i\in I}\Omega_i$ of X, such that Ω_i's are $3R$-separated, have diameters at most $f(3R)$ and $\mu(\Omega)\geq c\mu(X)$. Note that this means that $||P_\Omega\psi||_p^p=\mu(\Omega)\geq c\mu(X)=c||\psi||_p^p$.

Step 3: Norm estimates: With ψ and Ω as above, we have

$$\begin{aligned}||A\psi-AP_\Omega\psi||_p^p&\leq ||A||_p^p||\psi-P_\Omega\psi||_p^p=||A||_p^p\mu(X\setminus\Omega)=||A||_p^p(\mu(X)-\mu(\Omega))\leq\\ &\leq ||A||_p^p(1-c)\mu(X)\leq r^p(1-c)||\psi||_p^p.\end{aligned}$$

Consequently, we get

$$||AP_\Omega\psi||_p\leq ||A\psi||_p+r(1-c)^{1/p}||\psi||_p.$$

[1] If $p\neq 2$, we simply mean that the analogue of the Pythagoras' equality holds, i.e. that $||A\varphi||_p^p=\sum_{i\in I}||A\varphi_i||_p^p$.

Since the vector $P_\Omega\psi$ splits as $P_\Omega\psi = \sum_{i\in I} P_{\Omega_i}\psi$ (possibly discarding summands that are 0), where the summands have $3R > 2R$ separated supports, we can combine all of the above to obtain

$$\inf_{i\in I}\frac{\|A(P_{\Omega_i}\psi)\|_p}{\|P_{\Omega_i}\psi\|_p} \le \frac{\|AP_\Omega\psi\|_p}{\|P_\Omega\psi\|_p} \le \frac{\|AP_\Omega\psi\|_p}{c^{1/p}\|\psi\|_p} \le \frac{1}{c^{1/p}}\frac{\|A\psi\|_p}{\|\psi\|_p} + r\left(\frac{1-c}{c}\right)^{1/p}.$$

Also recall that $\mathrm{diam}(\mathrm{supp}(P_{\Omega_i}\psi)) \le f(3R)$. Hence we see that by choosing the vectors $\psi \ne 0$ such that $\frac{\|A\psi\|_p}{\|\psi\|_p}$ are arbitrarily close to $\nu(A)$, we can produce a another vector $\phi \ne 0$ (one of $P_{\Omega_i}\psi$'s) whose support has diameter at most $f(3R)$ and the fraction $\frac{\|A\phi\|_p}{\|\phi\|_p}$ is thus arbitrarily close to $\frac{\nu(A)}{c^{1/p}} + r(\frac{1-c}{c})^{1/p}$. Thus

$$\nu_{f(3R)}(A) \le \frac{\nu(A)}{c^{1/p}} + r(\tfrac{1-c}{c})^{1/p}.$$

Step 4: Replace c's by δ using that c can be chosen arbitrarily close to 1: Recall that once a space X has the Metric Sparsification Property for some $c \in (0,1]$, it has the Property for any $c \in (0,1)$ [2, Proposition 3.3]. Of course, making c bigger will possibly change the function f.

Since $0 \le \nu(A) \le \|A\|_p \le r$, for any $\delta > 0$ we can find $0 \le c < 1$, such that $\frac{\nu(A)}{c^{1/p}} + r(\frac{1-c}{c})^{1/p} \le \nu(A) + \delta$, since $c^{1/p} \to 1$ and $r(\frac{1-c}{c})^{1/p} \to 0$ as $c \to 1$. This finishes the proof.

Step 5: Incorporate the restrictions to $F \subset X$: The presented proof works exactly the same way, with the same constants, we only need to restrict the supports of the vectors ψ to the given F. □

Remark 4.2. The estimate in the Operator Norm Localisation Property [2, Definition 2.2] uses a multiplicative instead of an additive constant that appears in the formulation of Proposition 4.1. Perhaps there is a neater argument than the one above which would yield a multiplicative estimate (which should be then automatically independent of the bound on the norm of the operator A, the parameter r).

Remark 4.3. As finite asymptotic dimension [4] of X (say d) easily yields the Metric Sparsification Property with $c = \frac{1}{d+1}$, and quite often one also knows the function f associated with this c, the above proof makes it possible to be very explicit about the support bound D in these cases. This is in particular true for $\mathbb{Z}^d$ (we leave the computation to the reader as an exercise).

Remark 4.4. Proposition 4.1 fails for spaces X which does not have the Metric Sparsification Property. By [1] and [9], this is equivalent to not having the Operator Norm Localisation property, and so [8, Lemma 4.2] provides us with $R > 0$, $\varkappa < 1$, a sequence of disjoint finite subsets X_n of X, a sequence of positive, norm one operators $T_n \in \mathscr{B}(\ell^2 X_n)$ with propagation at most R and an increasing sequence of positive reals S_n tending to infinity, such that for any $\xi \in \ell^2 X_n$ of norm one, with support of diameter at most S_n, one has $\|T_n\xi\| \le \varkappa$. Furthermore, it is argued in [8, Proof of Theorem 1.3], that there are eigenvectors of T_n with eigenvalue 1.

Taking $N \geq 0$, and denoting $V_n = 1 - T_n \in \mathscr{B}(\ell^2 X_n)$, we see that $\nu_{S_N}(V_n) \geq 1 - \varkappa$ for all $n \geq N$, so the block-diagonal operator

$$Q_N = 1 \oplus_{n \geq N} V_n \in \mathscr{B}\left(\ell^2(X \setminus \sqcup_{n \geq N} X_n) \oplus_{n \geq N} \ell^2 X_n\right)$$

satisfies $\nu_{S_N}(Q_N) \geq 1 - \varkappa > 0$. However Q_N has a non-trivial kernel (as each V_n does), thus $\nu(Q_N) = 0$. Observe also that Q_N has norm one and propagation at most R. Thus if we choose $0 < \delta < 1 - \varkappa$, for any $D > 0$ we can take N sufficiently large, so that $S_N > D$ and so the operator Q_N will satisfy $\nu_D(Q_N) \geq \nu_{S_N}(Q_N) \geq 1 - \varkappa > 0 + \delta = \nu(Q_N) + \delta$. This violates Proposition 4.1.

Finally, we note that we can construct suitable V_n's explicitly under a slightly stronger assumption that X contains a disjoint union of finite subsets X_n, such that $\sqcup_n X_n$ is not uniformly locally amenable [1] (in particular, if $\sqcup_n X_n$ is an expander). Namely, we can take Laplacians $V_n = \Delta_R^{(n)} \in \mathscr{B}(\ell^2 X_n)$ on (a suitable) scale R, defined by

$$\Delta_R(\delta_x) = \sum_{y \in X_n, d(y,x) \leq R} (\delta_x - \delta_y), \qquad x \in X_n,$$

see [8, Section 3].

Coming back to the "positive side", and specialising to the case when $X = \Gamma$, a countable group endowed with a proper left-invariant metric, we conclude, as in [3, Corollary 7], that one can localise the lower norm uniformly for the whole of $\sigma_{\mathrm{op}}(A)$ for a fixed $T \in \mathscr{A}_p$:

Corollary 4.5. *Let $A \in \mathscr{A}_p$ and $\delta > 0$. Then there exists $D \in \mathbb{N}$, such that*

$$\nu(B|_F) \leq \nu_D(B|_F) \leq \nu(B|_F) + \delta$$

for all $F \subset X$ and $B \in \{A\} \cup \sigma_{\mathrm{op}}(A)$.

5. Proofs from [3]

For completeness, we recall the Theorem and the proof of [3, Theorem 8].

Theorem 5.1. *Let $A \in \mathscr{A}_p^{\$}$. Then there exists a $C \in \sigma_{\mathrm{op}}(A)$ with $\nu(C) = \inf\{\nu(B) \mid B \in \sigma_{\mathrm{op}}(A)\}$.*

Proof. Let $B_n \in \sigma_{\mathrm{op}}(A)$ be a sequence of operators, such that $\nu(B_n) \to \inf\{\nu(B) \mid B \in \sigma_{\mathrm{op}}(A)\}$. as $n \to \infty$. From each B_n, we construct E_n, a certain shift of B_n. The sequence $(E_n)_{n=1}^{\infty}$ (as any sequence in $\sigma_{\mathrm{op}}(A)$) has a subsequence converging $\mathscr{P}$-strongly to some $E \in \sigma_{\mathrm{op}}(A)$. For the sake of simpler notation, we assume that the sequence (E_n) itself converges $\mathscr{P}$-strongly to E.

Denote $\delta_i = \frac{1}{2^i}$ and $r_l = \sum_{i=l}^{\infty} \delta_i = \frac{1}{2^{l-1}}$, and note that $r_l \to 0$ as $l \to \infty$. The main property of E_n's is the following:

> For every l, there exists a finite set $F_l \subset \Gamma$, such that for any $n > l$ we have $\nu(E_n|_{F_l}) < \nu(B_n) + r_l$.

Having this property, we can finish the proof: by fixing l, we can restrict to (the finite set) F_l, which turns the $\mathscr{P}$-strong convergence $E_n \xrightarrow{\mathscr{P}-\text{strong}} E$ into a norm convergence $E_n|_{F_l} \xrightarrow{\|\cdot\|} E|_{F_l}$, and thus

$$\nu(E) \leq \nu(E|_{F_l}) = \lim_{n \to \infty} \nu(E_n|_{F_l}) \leq \lim_{n \to \infty} (\nu(B_n) + r_l) = \inf\{\nu(B) \mid B \in \sigma_{\mathrm{op}}(A)\} + r_l.$$

Taking now the limit $l \to \infty$, we obtain $\nu(E) \leq \inf\{\nu(B) \mid B \in \sigma_{\mathrm{op}}(A)\}$ and we are done.

Let us now describe the construction of E_n's. The lower norm localisation provides us for each $j \geq 0$ with a $D_j \geq 0$, such that $\nu_{D_j}(B|_F) < \nu(B|_F) + \delta_j$ for any $F \subset \Gamma$ and $B \in \sigma_{\mathrm{op}}(A)$. We can, and will, assume that $D_{j+1} > 2D_j$.

For the rest of the proof, we fix n. We shall be inductively constructing a sequence

$$C_0, C_1, \dots, C_n =: E_n$$

of shifts of B_n.

For the first step of the induction, apply the lower norm localisation estimate with δ_n and D_n to the operator B_n, giving a unit vector ζ with $\mathrm{diam}(\mathrm{supp}(\zeta)) \leq D_n$ and $\|B_n\zeta\| < \nu(B_n) + \delta_n$. It follows that there exists a shift U_{g_0}, such that $\xi_0 = U_{g_0}\zeta$ is supported on $B_e(D_n)$. Denote $C_0 = U_{g_0} B_n U_{g_0}^*$.

In loose terms, we repeat the same procedure, now with C_0, and with parameters δ_{n-1} (larger, so "worse") and D_{n-1} (smaller, so "better") to obtain another shift C_1 with a unit vector ξ_1 centred at e, now witnessing $\nu(B_n)$ up to $\delta_n + \delta_{n-1} < r_{n-1}$. During the actual induction, we also need to keep track of the total shift distance.

We proceed upwards from $k = 0$ to $k = n$, so the induction step is from k to $k+1$. Assume that we've constructed shifts $U_{g_0}, U_{g_1}, \dots U_{g_k}$ (so that $C_0 = U_{g_0} B_n U_{g_0}^*, C_1 = U_{g_1} C_0 U_{g_1}^*, \dots, C_k = U_{g_k} C_{k-1} U_{g_k}^*$) and unit vectors $\xi_0, \xi_1, \dots, \xi_k$ with the following properties:

(i) $\mathrm{supp}(\xi_i) \subset B_e(D_{n-i})$ for $i = 0, \dots, k$,
(ii) $\|C_i\xi_i\| < \nu(B_n) + r_{n-i}$ for $i = 0, \dots, k$ and
(iii) $|g_i| \leq D_{n-i+1}$ for $i = 1, \dots, k$.

Applying the lower norm localisation estimate with $\delta_{n-(k+1)}, D_{n-(k+1)}$ to the operator $C_k|_{B_e(D_{n-k})}$ yields a unit vector ζ supported on $B_e(D_{n-k})$ with $\mathrm{diam}\,\mathrm{supp}(\zeta) \leq D_{n-(k+1)}$ and $\|C_k\zeta\| < \nu(C_k|_{B_e(D_{n-k})}) + \delta_{n-(k+1)}$. Consequently, there exists a shift $U_{g_{k+1}}$ which "moves ζ to e", i.e. $\mathrm{supp}(U_{g_{k+1}}\zeta) \subset B_e(D_{n-(k+1)})$ and $|g_{k+1}| \leq D_{n-k}$ (iii). Denote $\xi_{k+1} = U_{g_{k+1}}\zeta$ (so that (i) holds) and $C_{k+1} = U_{g_{k+1}} C_k U_{g_{k+1}}^*$.

Note that (i) and the definition of ν imply that $\nu(C_k|_{B_e(D_{n-k})}) \leq \|C_k\xi_k\|$, so that we obtain the lower norm estimate (ii) for $i = k+1$ by computing:

$$\begin{aligned}\|C_{k+1}\xi_{k+1}\| = \|U_{g_{k+1}} C_k U_{g_{k+1}}^* U_{g_{k+1}}\zeta\| = \|C_k\zeta\| <\\ < \nu(C_k|_{B_e(D_{n-k})}) + \delta_{n-(k+1)} \leq \|C_k\xi_k\| + \delta_{n-(k+1)} <\\ < \nu(B_n) + r_{n-k} + \delta_{n-(k+1)} = \nu(B_n) + r_{n-(k+1)}.\end{aligned}$$

This finishes the induction.

It remains to show that we don't shift a particular ξ_k (which "works for a restriction of C_k" for a lower norm estimate) too much during the procedure of constructing $C_{k+1}, \dots, C_n$, so that we get a unit vector which "works for a restriction of C_n" as well.

Given $k \in \{0, \dots, n\}$, observe that the shifting distance of $U_{g_n} U_{g_{n-1}} \dots U_{g_{k+1}} = U_{g_n \cdots g_{k+1}}$ is at most $|g_n \cdots g_{k+1}| \leq D_1 + D_2 + \dots + D_{n-k} \leq 2D_{n-k}$, so that

$$\mathrm{supp}(U_{g_n} U_{g_{n-1}} \dots U_{g_{k+1}} \xi_k) \subset B_e(3D_{n-k}).$$

Since

$$C_n = U_{g_n} C_{n-1} U_{g_n}^* = \cdots = U_{g_n} U_{g_{n-1}} \ldots U_{g_{k+1}} C_k U_{g_{k+1}}^* \ldots U_{g_n}^*,$$

we see that

$$\nu(C_n|_{B_e(3D_{n-k})}) \leq ||C_n(U_{g_n} U_{g_{n-1}} \ldots U_{g_{k+1}} \xi_k)|| = ||C_k \xi_k|| \leq \nu(B_n) + r_{n-k}.$$

Denoting $E_n = C_n$, $l = n - k$, $F_l = B_e(3D_l) \subset \Gamma$, we obtain

$$\nu(E_n|_{F_l}) \leq \nu(B_n) + r_l \qquad \text{for } l \leq n,$$

which finishes the proof. □

We close the piece with an open problem: While the main tool, Proposition 4.1, holds precisely for spaces with the Metric Sparsification Property (i.e. exact groups in the group case), the "Big Question" itself is still open for non-exact groups. More precisely, can one give an example of an operator $A \in \mathscr{A}_p$, necessarily for a non-exact group Γ, such that all limit operators $B \in \sigma_{\mathrm{op}}(A)$ are invertible, but without a uniform bound on the norms of inverses $||B^{-1}||$? This is equivalent to asking for an operator $A \in \mathscr{A}_p$, which is not invertible modulo the ideal of ghost operators, but for which all limit operators are invertible.

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