On the locally compact case of Toruńczyk's Approximation Theorem

(Master thesis)

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Contents

Introduction

In the infinite–dimensional topology, Toruńczyk's approximation theorem is one of the strong results. It is the major step in proving the characterization of the Hilbert Cube. It has been proved first in 1978 by H. Toruńczyk, but the proof used another strong result. Thereafter, in 1979, R. D. Edwards published a different, more "elementary" proof in [Ed]. However, the published proof is quite brief and focused onto compact case. Next, in 1989, more detailed version of Edward's proof for compact case was published in J. van Mill's book [vM]

In this thesis we focus on the details of Edwards's proof of the Toruńczyk's approximation theorem, but now in full generality, i.e. for the locally compact case.

Most of the statements stated in this thesis are taken from the book [vM]. We follow the exposition there closely; so in the parentheses after the label of each statement, there's usually a number of corresponding statement in the book. The number with an asterisk * refer to the statement in [vM] that is being generalized.

Chapter 1

Statements taken from books; prerequisites

We suppose all the spaces in this thesis are separable metric (with the only exception of the space $\mathcal{C}(X,Y)$ for noncompact X and Y).

We sometimes write "iff" instead of "if and only if". Furthermore, we denote the interval $[0,1]$ by I and $[-1,1]$ by J throughout the text.

Let X and Y be spaces. We denote by $\mathcal{C}(X,Y)$ the set of all continuous functions from X to Y; by $\mathcal{H}(X,Y) \subset \mathcal{C}(X,Y)$ set of all homeomorphisms from X to Y and we put $\mathcal{H}(X) = \mathcal{H}(X,X)$. If $\mathcal{V}$ is an open covering of Y, we say that two maps $f, g : X \to Y$ are *$\mathcal{V}$-close* if for each $x \in X$ there exists $V \in \mathcal{V}$ such that $f(x), g(x) \in V$. If $\mathcal{U}$ is an open cover of X, we say that the map $f : X \to Y$ is *a $\mathcal{U}$-map* if for every $y \in Y$ there is $U \in \mathcal{U}$ such that $f^{-1}(y) \subset U$. For maps $f, g : X \to Y$ we write $f \sim g$ if f and g are homotopic (i.e. there exists a continuous map $H : X \times I \to Y$ with $H \restriction X \times \{0\} = f$ and $H \restriction X \times \{1\} = g$; we call H *a homotopy* and use the notation $H_t = H \restriction X \times \{t\}$). We say that the homotopy $H : X \times I \to Y$ is *limited by an open cover $\mathcal{V}$ of Y* if for every $x \in X$ there is $V \in \mathcal{V}$ such that $H(\{x\} \times I) \subset V$. In this case we also call H *a $\mathcal{V}$-homotopy.* If two maps $f, g : X \to Y$ are homotopic by a homotopy which is limited by an open cover $\mathcal{V}$ of Y, we write $f \sim_{\mathcal{V}} g$. If $H : X \times I \to X$ is a homotopy such that each level (i.e. the mapping $H_t : X \to X$ for $t \in I$) of it is a homeomorphism, we call such H *an isotopy.*

1.1 Proper and closed maps

We shall use throughout the text, without any special reference, the following simple fact about closed maps:

Proposition 1.1. *Let $f : X \to Y$ be a closed continuous map between spaces X and Y. Moreover, let A be subset of Y and let U be an open neighborhood of*

$f^{-1}(A)$ in X. Then there exists an open neighborhood V of A in Y such that $f^{-1}(V) \subset U$.

Proof. Under the assumptions of the proposition, it follows that $X - U$ is closed in X, therefore $f(X - U)$ is closed in Y, so $Y - f(X - U)$ is open. Denote this set by V. We prove that this set is as needed.

If $a \in A$, then $f^{-1}(a) \subset f^{-1}(A) \subset U$; so $f^{-1}(a) \cap (X - U) = \emptyset$. It follows that $a \in Y - f(X - U) = V$. So we just proved $A \subset V$.

Suppose that $x \in f^{-1}(V) = f^{-1}(Y - f(X - U))$. Consequently, $f(x) \in Y - f(X - U)$, so $f(x) \notin f(X - U)$. Therefore $x \notin X - U$, so $x \in U$. Summarizing, $f^{-1}(V) \subset U$. We are done. □

A continuous map $f : X \to Y$ between locally compact spaces X and Y is

- *proper* provided that $f^{-1}(K)$ is compact for each compact $K \subset Y$,
- *perfect* provided that it's closed and $f^{-1}(y)$ is compact for each $y \in Y$.

We'll make use of these concepts primarily in the "onto" case. Notice that the finite composition of proper maps is again proper, this is trivial.

Proposition 1.2. *A continuous surjection $f : X \to Y$ between locally compact spaces X and Y is proper if and only if it is perfect.*

Proof. For the implication "$\Longrightarrow$" it's enough to show that f is closed. Take closed $F \subset X$ and a sequence of points $y_n \in f(F)$ converging to a point $y \in Y$. By assumption, the pre–image A of compact set $\{y_n \mid n \in \mathbb{N}\} \cup \{y\}$ is compact. The set $f^{-1}(y_n) \cap F \subset A$ is nonempty for every n, so we can choose a point from it, say x_n. By compactness of A we have that there's a subsequence x_{i_j}, $j \in \mathbb{N}$ converging to, say $x \in X$. But closedness of F implies $x \in F$. Since f is continuous, we have $y_{i_j} = f(x_{i_j}) \to f(x)$, but we know that y_{i_j} converges also to y, so $y = f(x) \in f(F)$. So $f(F)$ is closed.

For the converse, take compact set $C \subset Y$, and take any sequence $(x_n)_{n\in\mathbb{N}} \subset f^{-1}(C)$. Then $(f(x_n))_{n\in\mathbb{N}} \subset C$ has a cluster point, say $y \in C$. We show that in any neighborhood U of the fiber $f^{-1}(y)$ there are infinitely many x_n's. By closedness of f we have a neighborhood V of y in Y with $f^{-1}(V) \subset U$. But in any such V there are infinitely many $f(x_n)$'s, therefore these $x_n \in f^{-1}f(x_n) \subset f^{-1}(V) \subset U$.

Now suppose that $(x_n)_{n\in\mathbb{N}}$ does not have a cluster point. Then for every $z \in f^{-1}(y)$ exists a neighborhood U_z of z containing only finitely many x_n's. Such U_z's cover the compact set $f^{-1}(y)$, so there's a finite subcover, say $U_{z_1}, \ldots, U_{z_k}$. But then $U_{z_1} \cup \cdots \cup U_{z_k}$ is the neighborhood of $f^{-1}(y)$ containing only finitely many x_n's. Contradiction. So every sequence in $f^{-1}(C)$ has a cluster point, hence $f^{-1}(C)$ is compact. □

Proposition 1.3. *Let X and Y be locally compact spaces, $f : X \to Y$ be a proper surjection. Then there exists an open cover $\mathcal{U}$ of Y such that for any space Z and continuous maps $g, h : Z \to X$ which are $f^{-1}(\mathcal{U})$-close we have g is proper iff h is proper. In particular, for any locally compact space X there is an open cover $\mathcal{W}$ of X such that for any space Z, two $\mathcal{W}$-close maps are either both proper or both not.*

Proof. First remark that "in particular" part follows from the former statement by letting $f = 1_X$.

We write $Y = \bigcup_{n\in\mathbb{N}} K_n$, where K_n is compact and $K_n \subset \operatorname{int} K_{n+1}$ for every $n \in N$. This is possible, because Y, being locally compact separable metric space, has the base of topology consisting of sets with compact closures. Put

$$\mathcal{U} = \{\operatorname{int}(K_{n+2}) - K_n \mid n \in \mathbb{N}\} \cup \{\operatorname{int}(K_2)\}.$$

We show that this is as required.

Denote also $C_n = f^{-1}(K_n)$ for every $n \in \mathbb{N}$. Since f is proper, we see that each C_n is compact. For any set $A \subset Y$, $f^{-1}(\operatorname{int}(A))$ is an open set contained in $f^{-1}(A)$, so also in $\operatorname{int}(f^{-1}(A))$, therefore we have

$$f^{-1}(\operatorname{int}(K_{n+2}) - K_n) = f^{-1}(\operatorname{int}(K_{n+2})) - f^{-1}(K_n) \subseteq \operatorname{int}(C_{n+2}) - C_n$$

and

$$C_n = f^{-1}(K_n) \subseteq f^{-1}(\operatorname{int}(K_{n+1})) \subseteq \operatorname{int}(f^{-1}(K_{n+1})) = \operatorname{int}(C_{n+1}).$$

Moreover, $X = \bigcup_{n\in\mathbb{N}} C_n$. Summarizing, the open cover $\mathcal{V} = \{\operatorname{int}(C_{n+2}) - C_n \mid n \in \mathbb{N}\} \cup \{\operatorname{int}(C_2)\}$ of X is such that $f^{-1}(\mathcal{U})$ refines it. So it suffices to prove that if a proper map $g : Z \to X$ and a continuous map $h : Z \to X$ are $\mathcal{V}$-close, then also h is proper.

So let $C \subset X$ be compact. Then there is a finite subcover of $\mathcal{V}$ covering it, so there is $k \in \mathbb{N}$ with $C \subset \operatorname{int}(C_k)$. If $x \in h^{-1}(C)$, then $h(x) \in C \subset \operatorname{int}(C_k)$. But since g and h are $\mathcal{V}$-close, we have that $g(x) \in \operatorname{int}(C_{k+1})$. Therefore $h^{-1}(C) \subset g^{-1}(C_{k+1})$, and since $g^{-1}(C_{k+1})$ is compact, $h^{-1}(C)$, being its closed subset, is also compact. It follows that also h is proper, what finishes the proof. □

1.2 Topologies on some spaces

Let X and Y be locally compact spaces. We topologize the space $\mathcal{C}(X, Y)$ of all continuous functions from X to Y by letting the basic neighborhood of any $f \in \mathcal{C}(X, Y)$ be all sets of the form $N(f, \mathcal{U})$ for any open cover $\mathcal{U}$ of Y, where

$$N(f, \mathcal{U}) = \{g \in \mathcal{C}(X, Y) \mid f \text{ and } g \text{ are } \mathcal{U}\text{-close}\}.$$

This is clearly the same as to say that basic neighborhoods of f are of the form $N(f, \epsilon)$, for any $\epsilon : Y \to (0, \infty)$, where (if d is an admissible metric on Y)

$$N(f, \epsilon) = \{g \in \mathcal{C}(X, Y) \mid d(f(x), g(x)) < \epsilon(x) \text{ for all } x \in X\}.$$

Another equivalent description of this topology is as follows: it's induced by the family

$$\{\hat{d} \mid d \text{ is an admissible bounded metric on Y}\}$$

of metrics on $\mathcal{C}(X,Y)$, where

$$\hat{d}(f,g) = \sup_{x \in X} d(f(x), g(x))$$

for $f, g \in \mathcal{C}(X,Y)$ (see [Be], page 121). The metric $\hat{d}$ is complete iff d is complete (see [vM], corollary 1.3.5). Notice that if X is compact space, the space $\mathcal{C}(X,Y)$ is separable and metrizable with any of metrics $\hat{d}$, since they are all equivalent (see [vM], section 1.3).

It follows easily from proposition 1.3 that all the proper maps in $\mathcal{C}(X,Y)$ form an open–closed subset of the space $\mathcal{C}(X,Y)$.

Proposition 1.4. *Let X and Y be locally compact spaces, $K \subset X$ be closed and $h : X \to Y$ be proper map. Then the set $\mathcal{A} = \{g \in \mathcal{C}(X,Y) \mid g \restriction K = h \restriction K$ and g is proper$\}$ with the subspace topology is Baire space.*

Proof. We are to prove that the intersection of the countable family $(U_i)_{i \in \mathbb{N}}$ of dense open sets in $\mathcal{A}$ is dense in $\mathcal{A}$. Fix $f \in \mathcal{A}$. We show that any neighborhood of f in $\mathcal{A}$ has nonempty intersection with $\bigcap_{i \in \mathbb{N}} U_i$.

Since Y is locally compact, it is also complete. Consequently, for any neighborhood V of f in $\mathcal{C}(X,Y)$ there is a complete metric d on Y such that $U = \{g \in \mathcal{C}(X,Y) \mid \hat{d}(g,f) \leq 1\} \subseteq V$ (it follows from the characterization of the topology on $\mathcal{C}(X,Y)$). Observe that also $\hat{d}$ is complete and since proper maps comprise an closed subset of $\mathcal{C}(X,Y)$ and the set $\{g \in \mathcal{C}(X,Y) \mid g \restriction K = h \restriction K\}$ is also closed in $\mathcal{C}(X,Y)$, it follows that $\hat{d}$ is complete also on $\mathcal{A}$. Now we construct a $\hat{d}$-Cauchy sequence f_i in $\mathcal{A}$ that converges to $g_0 \in (U \cap \mathcal{A}) \cap \bigcap_{i \in \mathbb{N}} U_i$. But that means that $(V \cap \mathcal{A}) \cap \bigcap_{i \in \mathbb{N}} U_i \neq \emptyset$, hence f is in closure of $\bigcap_{i \in \mathbb{N}} U_i$ in $\mathcal{A}$.

What remains is to show how we construct such a sequence of f_i's. This is done similarly as in the proof of Baire theorem. By induction we construct a sequence of positive reals ϵ_i and continuous functions $f_i \in \mathcal{A}$ such that (for all positive integers i)

- $\hat{d}(f_i, f_{i+1}) < \epsilon_i < 1/2^{i-1}$,
- $\overline{B(f_{i+1}, \epsilon_{i+1})} \subset B(f_i, \epsilon_i) \cap U_{i+1}$,
- $B(f_i, \epsilon_i) \subset U \cap U_1 \cap \cdots \cap U_i$,

where $B(g, \delta)$ denotes an $\hat{d}$–open ball around g with diameter δ in $\mathcal{A}$. This can by easily done since each U_i is open and dense in $\mathcal{A}$. It follows that such sequence is Cauchy, by completeness of $\hat{d}$ it has a limit, this limit point is in the closure of every $B(f_i, \epsilon_i)$, so it is in each U_i and also in $U \cap \mathcal{A}$. This completes the proof. □

Now we define *the mapping cylinder* $M(f)$ of the proper surjection $f : X \to Y$ between locally compact spaces X and Y. (We stick to this notation for the rest of the section.) We put $M(f) = X \times [0,1) \cup Y$ (we think of $X \times [0,1)$ and Y being disjoint) endowed following topology:

- on $X \times [0,1)$ we put usual product topology,
- we let basic neighborhoods of $y \in Y$ be of the form $f^{-1}(U) \times (t,1) \cup U$ for any $t \in [0,1)$ and any open $U \subset Y$ containing y.

It is sometimes convenient to think of the mapping cylinder as on the picture 1.1. Next define the mapping $\pi_f : X \times I \to M(f)$ by

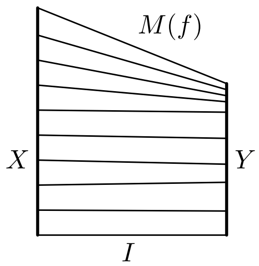


Figure 1.1: The mapping cylinder

$$\pi_f(x,t) = \begin{cases} (x,t) & \text{for } t \in [0,1), \\ f(x) & \text{for } t = 1. \end{cases}$$

Notice that since f is surjective, also π_f is surjective.

Proposition 1.5. *For locally compact spaces X and Y and a proper surjection $f : X \to Y$ the topology on mapping cylinder $M(f)$ is the same as quotient topology with $\pi_f : X \times I \to M(f)$ as a quotient map.*

Proof. We are to prove that for any $V \subset M(f)$

$$\pi_f^{-1}(V) \text{ is open in } X \times I \iff V \text{ is open in } M(f).$$

Take any $V \subset M(f)$.

First handle the implication "$\Longleftarrow$". If $V \subset X \times [0,1)$, then there's nothing to prove, since $\pi_f \restriction X \times [0,1)$ is an identity. So it's enough to prove it for the sets of the type $f^{-1}(U) \times (t,1) \cup U$ for $U \subset Y$ open in Y. But then $\pi_f^{-1}(f^{-1}(U) \times (t,1) \cup U) = f^{-1}(U) \times (t,1]$ and this set is clearly open in $X \times I$.

Now the implication "$\Longrightarrow$". Again, if $V \subset X \times [0,1)$, there's nothing to prove. So let $V \cap Y \neq \emptyset$. It's enough to show that for every $y \in Y$ there is a basic neighborhood of y in $M(f)$ which is contained in V.

Since f is proper, $\pi_f^{-1}(y) = f^{-1}(y) \times \{1\}$ is compact set contained in the open set $\pi_f^{-1}(V)$ in $X \times I$. Consequently, there exists $\epsilon > 0$ and an open $W \subset X$ such that $f^{-1}(y) \subset W$ and $W \times (\epsilon, 1] \subset \pi_f^{-1}(V)$. By closedness of f we obtain an open neighborhood U in Y of y with $f^{-1}(U) \subset W$. But then clearly $y \in f^{-1}(U) \times (\epsilon, 1) \cup U \subset V$. We are done. □

Proposition 1.6. *For locally compact spaces X and Y and proper surjection $f : X \to Y$ the mapping cylinder $M(f)$ is separable metrizable space.*

Proof. This is clear, because the topology we defined on $M(f)$ is obviously second countable, regular and T_1. □

We give the definition of *the collapse to the base* mapping. It is the function $c(f) : M(f) \to Y$ defined by letting

$$\begin{cases} c(f)(y) = y & \text{for } y \in Y, \\ c(f)(x,t) = f(x) & \text{for } x \in X, 0 \le t < 1. \end{cases}$$

It is obviously continuous and its point-inverses are contractible. Notice that since f is surjective, also $c(f)$ is surjective.

We finish the discussion of the mapping cylinders by observing that if f is proper, also π_f and $c(f)$ are proper. To see this, consider the following commutative diagram (where *proj* : $X \times I \to X$ denotes the projection):

$$\begin{array}{ccc} X \times I & \xrightarrow{proj} & X \\ \downarrow{\scriptstyle \pi_f} & & \downarrow{\scriptstyle f} \\ M(f) & \xrightarrow{c(f)} & Y. \end{array}$$

Now, if $A \subset Y$ is compact, then $c(f)^{-1}(A) = \pi_f(proj^{-1}(f^{-1}(A)))$ is also compact by properness of f and the projection. Therefore, $c(f)$ is proper.

If $B \subset M(f)$ is compact, then $\pi_f^{-1}(B)$ is closed subset of $proj^{-1}(f^{-1}(c(f)(B)))$, which is clearly compact. So also π_f is proper.

1.3 Simplicial complexes

Lemma 1.7 (3.6.6). *Let $\mathcal{T}$ be a simplicial complex and let X be a space. A function $f : |\mathcal{T}| \to X$ is continuous iff the restriction of f to every simplex $\tau \in \mathcal{T}$ is continuous on τ.*

Theorem 1.8 (remark after 3.6.12). *For every polytope $|\mathcal{T}|$ and for every open cover $\mathcal{U}$ of $|\mathcal{T}|$ there exists a subdivision $\mathcal{S}$ of $\mathcal{T}$ such that each simplex $\sigma \in \mathcal{S}$ is contained in an element of $\mathcal{U}$.*

1.4 ANR's

Theorem 1.9 (5.1.1). *Let X be a ANR. Then for every open cover $\mathcal{U}$ of X there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that for any space Y, any two $\mathcal{V}$-close continuous maps $f, g : Y \to X$ are $\mathcal{U}$-homotopic.*

Lemma 1.10 (5.1.7). *Let X be a space and let $\mathcal{U}$ be an open cover of X. Then there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is star–finite and star–refinement of $\mathcal{U}$.*

Theorem 1.11 (5.1.8). *Let X be an ANR. Then for every open cover $\mathcal{U}$ of X there exists a polytope P such that P $\mathcal{U}$-dominates X.*

1.5 The Hilbert Cube

We denote $\mathcal{Q} = J^{\mathbb{N}}$ (we regard this as a topological power). The topology on $\mathcal{Q}$ can be generated for example by the metric $d_{\mathcal{Q}}(x, y) = \max_{n\in\mathbb{N}} \frac{1}{2^n}|x_n - y_n|$.

Theorem 1.12 (6.4.6). *Let $A, B \in \mathcal{Z}(\mathcal{Q})$ and $f : A \to B$ be a homeomorphism with $d(f, 1_{\mathcal{Q}}) < \epsilon$ for some $\epsilon > 0$. Then there is a homeomorphism $\bar{f} : \mathcal{Q} \to \mathcal{Q}$ extending f such that $d(\bar{f}, 1_{\mathcal{Q}}) < \epsilon$.*

1.6 Bing Shrinking Criterion

Let X and Y be locally compact spaces. We say that a continuous surjection $f : X \to Y$ is *shrinkable* provided that for every open covers $\mathcal{U}$ of X and $\mathcal{V}$ of Y there exists a homeomorphism $h : X \to X$ satisfying

- $f \circ h^{-1}$ is $\mathcal{U}$-map, i.e. for every $y \in Y$ there is $U \in \mathcal{U}$ with $h(f^{-1}(y)) \subset U$,
- f and fh are $\mathcal{V}$-close.

We say that the continuous surjection $f : X \to Y$ is a *near homeomorphism*, if f is in the closure of the $\mathcal{H}(X, Y)$ in $\mathcal{C}(X, Y)$, i.e. for every open cover $\mathcal{V}$ of Y there is a homeomorphism $g : X \to Y$ such that f and g are $\mathcal{V}$-close.

Theorem 1.13 (Bing Shrinking Criterion; 6.1.8*). *Let X and Y be locally compact spaces and let $f : X \to Y$ be a proper surjection. Then f is a near homeomorphism iff f is shrinkable.*

In the proof, we make use of Bing Criterion for compact spaces; this is the theorem 6.1.8 in [vM]. We shall use the notation $(f, g)_{\mathcal{V}}$ if the maps $f, g : X \to Y$ are $\mathcal{V}$-close for the open cover $\mathcal{V}$ of Y.

Proof. Assume that $f : X \to Y$ is a near homeomorphism. We prove that it's also shrinkable. Fix $\mathcal{U}$ and $\mathcal{V}$. Let $\mathcal{V}_1$ be a star-refinement of $\mathcal{V}$ (lemma 1.10). Since we assume that f is a near homeomorphism, there is a homeomorphism $p : X \to Y$ with $(f, p)_{\mathcal{V}_1}$. Now let $\mathcal{V}_2$ be the star-refinement of the common open refinement of $\mathcal{V}_1$ and the open cover $p(\mathcal{U})$ (lemma 1.10). Denote by $q : X \to Y$ a homeomorphism with $(f, q)_{\mathcal{V}_2}$ and let $h = p^{-1}q$. We claim that h is required shrinking homeomorphism.

To see that $(f, fh)_{\mathcal{V}}$, consider the following sequences of implications:

$$(p, f)_{\mathcal{V}_1} \implies (1_Y, fp^{-1})_{\mathcal{V}_1} \implies (q, fp^{-1}q)_{\mathcal{V}_1},$$
$$(f, q)_{\mathcal{V}_2} \implies (f, q)_{\mathcal{V}_1}.$$

Since $\mathcal{V}_1$ is star-refinement of $\mathcal{V}$, it follows that $(f, fp^{-1}q)_{\mathcal{V}}$, so f and fh are $\mathcal{V}$-close.

It remains to show that $fh^{-1} = fq^{-1}p$ is $\mathcal{U}$-map. To this end, fix $y \in Y$. For any $x \in f^{-1}(y)$ we have that there exists a $V_x \in \mathcal{V}_2$ such that it contains both $y = f(x)$ and $q(x)$. So all of the V_x's for $x \in f^{-1}(y)$ contain y, and since $\mathcal{V}_2$ is star-refinement of $p(\mathcal{U})$, it follows that there is $U \in \mathcal{U}$ such that $q(x) \in p(U)$ for all $x \in f^{-1}(y)$. In other words, $q(f^{-1}(y)) \subset p(U)$, so $p^{-1}(q(f^{-1}(y))) \subset U$, for some $U \in \mathcal{U}$. This finishes the argument.

For the converse implication, assume that $f : X \to Y$ is shrinkable and fix an open cover $\mathcal{V}$ of Y. We do the "1-point compactification" trick, as done in [Ch]. Assume that we're dealing with non-compact X and Y. Denote $\tilde{X} = X \cup \{\infty\}$ and $\tilde{Y} = Y \cup \{\infty\}$ the one-point compactifications of X and Y. Extend f to $\tilde{f} : \tilde{X} \to \tilde{Y}$ by letting $\tilde{f}(\infty) = \infty$. It's clear that $\tilde{f}$ is proper surjection and moreover, it's shrinkable (we can use the same shrinking homeomorphisms h for f, just extend them to $\tilde{h}$ by putting $\tilde{h}(\infty) = \infty$). The proof of the compact case yields the homeomorphism $g : \tilde{X} \to \tilde{Y}$ which is as close to $\tilde{f}$ as we wish — i.e. we can choose it $(\{\tilde{Y}\} \cup \mathcal{V})$-close to $\tilde{f}$. It is clear from the proof of the compact case that we can arrange that $g(\infty) = \infty$. If we denote by $\bar{g}$ the restriction of g to X, we see that indeed $\bar{g} : X \to Y$ is the homeomorphism which is $\mathcal{V}$-close to f. $\square$

Chapter 2

Generalized statements

2.1 Fine homotopy equivalences, cell–like maps

A compactum X has *trivial shape* if for each embedding f of X into an ANR Z, $f(X)$ is contractible in any of its neighborhoods. Let X and Y be locally compact spaces and $f : X \to Y$ be a continuous map. We say f is a *fine homotopy equivalence* if for each open cover $\mathcal{U}$ of Y there exists a continuous map $g : Y \to X$ such that $g \circ f : X \to X$ is $f^{-1}(\mathcal{U})$-homotopic to the identity on X and $f \circ g : Y \to Y$ is $\mathcal{U}$-homotopic to the identity on Y. Moreover if f is proper, we define analogously *a proper fine homotopy equivalence*, in the class of proper maps. Furthermore, we say that proper f is *cell–like* if it is onto and $f^{-1}(y)$ has trivial shape for each $y \in Y$.

In order to give a proof of Haver's theorem (which says that in the class of the locally compact ANR's, cell–like maps are exactly proper fine homotopy equivalences), we need to formulate the following two statements. Proofs are given in the book [vM]. First one is just reformulation of the property "having a trivial shape".

Theorem 2.1 (7.1.1). *Let X be a compact space and let Y be an ANR containing X. Then the following statements are equivalent:*

(a) X has trivial shape,

(b) if U is a neighborhood of X in Y then X is contractible in U.

Lemma 2.2 (7.1.4). *Let X be an ANR and let $f : X \to Y$ be cell–like. Then for every $y \in Y$ and for every neighborhood U of y in Y there exists a neighborhood $V \subset U$ of y in Y such that $f^{-1}(V)$ is contractible in $f^{-1}(U)$.*

The next statement is the key tool in the proof of Haver's theorem. The idea of the proof is taken from [Ha]. For the simplicial complexes, we will not distinguish between simplex and its geometrical realization.

Proposition 2.3 (7.1.5*). *Let X be a locally compact ANR and let $f : X \to Y$ be cell–like. Moreover, let K be a polytope, $L \subseteq K$ be a subpolytope, and let $\phi : K \to Y$ and $\psi' : L \to X$ be continuous mappings with $f \circ \psi' = \phi \restriction L$. Then for every open cover $\mathcal{U}$ there is a continuous function $\psi : K \to X$ extending ψ' such that $f \circ \psi$ and ϕ are $\mathcal{U}$-close.*

Proof. First fix some notation. For locally finite simplicial complex $\mathcal{T}$ and any $\sigma \in \mathcal{T}$ we denote $N(\sigma, \mathcal{T}) = \{\tau \in \mathcal{T} \mid \sigma \cap \tau \neq \emptyset\}$, $\mathrm{st}(\sigma, \mathcal{T}) = \{\tau \in \mathcal{T} \mid \sigma \subseteq \tau\}$ and finally ${}^j\mathcal{T} = \{\sigma \in \mathcal{T} \mid |N(\sigma, \mathcal{T})| \subset |\mathcal{T}^{(j)}|\}$ for any integer $j \geq 0$ (this is the part of $\mathcal{T}$, which is at most j-dimensional in a sense that all it "touches" is at most j-dimensional).

Now we inductively construct a sequence $\mathcal{U}_0 > \mathcal{U}_1 > \dots$ of open covers of Y and a sequence $K_0, K_1, \dots$ of triangulations of K with the following properties:

(1) $\mathcal{U}_0 < \mathcal{U}$ and for every integer $i \geq 1$ and every $U \in \mathcal{U}_i$ there is $V \in \mathcal{U}_{i-1}$ such that $f^{-1}(\mathrm{St}(U, \mathcal{U}_i))$ is contractible in $f^{-1}(V)$ (this is possible: first we refine $\mathcal{U}_{i-1}$ according to lemma 2.2, then we let $\mathcal{U}_i$ be the star–refinement of this (lemma 1.10)),

(2) for every integer $i \geq 1$ let K_i be a subdivision of K_{i-1} subordinated to $\phi^{-1}(\mathcal{U}_i)$ (we use theorem 1.8).

Now we define inductively sequence of mappings $\psi_i : K_i^{(i)} \to X$, $i \geq 0$, with the following properties (for every integer $i \geq 0$):

(3) ψ_i agrees with ψ' on $K_i^{(i)} \cap L$ and with ψ_{i+1} on ${}^iK_i$,

(4) if σ is a j-simplex in $K_i^{(i)}$ and $k = \dim(\mathrm{st}(\sigma, K_i^{(i)}))$, then there is $U \in \mathcal{U}_{k-j}$ with $f(\psi_i(\sigma)) \cup \phi(\sigma) \subset U$,

(5) for each vertex $v \in K_i$, $\psi_i(v) \in f^{-1}(\phi(v))$.

We show that we can define $\psi_0 : K_0^{(0)} \to X$. If σ is 0-simplex of K_0 then σ is just a vertex with $\dim(\mathrm{st}(\sigma, K_0)) = 0$. So we put $\psi_0(\sigma) = \psi'(\sigma)$ if $\sigma \in L$, else we let $\psi_0(\sigma)$ be any point in $f^{-1}(\phi(\sigma))$. Then conditions (3) and (5) are clearly satisfied and (4) holds since $f(\psi_0(\sigma)) \cup \phi(\sigma) = \phi(\sigma)$ and $\mathcal{U}_0$ is an open cover of Y.

So suppose we already have ψ_i defined and we are to define ψ_{i+1}. We define it on each simplex of $K_{i+1}^{(i+1)}$ by induction on the dimension, then invoke lemma 1.7 for continuity. (To be more precise, for every simplex σ we define continuous function $\psi_\sigma : \sigma \to X$ and at the end we let ψ_{i+1} be the union of all ψ_σ's. But we abuse the notation and write ψ_{i+1} only.)

First define ψ_{i+1} on vertices $v \in K_{i+1}^{(i+1)}$. If $v \in L$, we put $\psi_{i+1}(v) = \psi'(v)$; if $v \in {}^iK_i$, we put $\psi_{i+1}(v) = \psi_i(v)$ (since $\dim(\mathrm{st}(v, K_i^{(i)})) = \dim(\mathrm{st}(v, K_{i+1}^{(i+1)}))$, we see that condition (4) is satisfied by inductive hypothesis). In the remaining case let $\psi_{i+1}(v)$ be any point in $f^{-1}(\phi(\sigma))$. Again since $f(\psi_{i+1}(v)) \cup \phi(v) = \phi(v)$

is a point, there clearly exists $U \in \mathcal{U}_k$ containing it, so (4) is again satisfied. Conditions (3) and (5) obviously hold.

Suppose we have defined ψ_{i+1} on all simplices of $K_{i+1}^{(i+1)}$ of dimension at most j and take any $(j+1)$-simplex $\sigma \in K_{i+1}^{(i+1)}$. If $\sigma \subset L$, we put $\psi_{i+1} \restriction \sigma = \psi' \restriction \sigma$. If $\sigma \subset {}^iK_i$, we put $\psi_{i+1} \restriction \sigma = \psi_i \restriction \sigma$. In the remaining case let $k = \dim(\mathrm{st}(\sigma, K_{i+1}^{(i+1)}))$. Observe that $i+1 \geq k-j$. Since K_{i+1} is subordinated to $\phi^{-1}(\mathcal{U}_{i+1}) < \phi^{-1}(\mathcal{U}_{k-j})$, there is $U \in \mathcal{U}_{k-j}$ containing $\phi(\sigma)$. For every proper face τ of σ of dimension j we have $f(\psi_{i+1}(\tau)) \cup \phi(\tau) \subset U'$ for some $U' \in \mathcal{U}_{k-j}$, by inductive hypothesis (4). We have $\phi(\tau) \subset \phi(\sigma)$, therefore $U \cap U' \neq \emptyset$, so $f(\psi_{i+1}(\tau)) \subset U' \subset \mathrm{St}(U, \mathcal{U}_{k-j})$. But τ was arbitrary such face, so we find that $f(\psi_{i+1}(\partial\sigma)) \cup \phi(\sigma) \subset \mathrm{St}(U, \mathcal{U}_{k-j})$. By the construction of $\mathcal{U}_{k-j}$ there exists $V \in \mathcal{U}_{k-j-1}$ such that $f^{-1}(\mathrm{St}(U, \mathcal{U}_{k-j}))$ is contractible in V. By inductive hypothesis ψ_{i+1} is defined on $\partial\sigma$ and now we can extend it continuously to the interior of σ with $\psi_{i+1}(\sigma) \subset V \in \mathcal{U}_{k-j-1}$ (because $\partial\sigma \approx S^j$, $\sigma \approx B^{j+1}$ and $f^{-1}(\mathrm{St}(U, \mathcal{U}_{k-j}))$ is contractible in V). We see that condition (3) is satisfied, and also (4) holds now for σ. This finishes the induction.

Observe now, that since K is polytope, so locally finite simplicial complex, for every point $x \in K$ there is a simplex σ and $N \in \mathbb{N}$ such that $x \in \sigma \subset {}^NK_N$. This means that $\psi_i(x) = \psi_N(x)$ for all integers $i \geq N$. Hence the function $\psi = \lim_{i\to\infty} \psi_i$ is well–defined and continuous from K to X and extends ψ'. Notice that from (4) it follows that there exists $U \in \mathcal{U}_i$ with $f(\psi(\sigma)) \cup \phi(\sigma) \subset U$, for some integer i. Since $\mathcal{U}_i$ refines $\mathcal{U}$, we see that $f \circ \psi$ and ϕ are $\mathcal{U}$-close. This finishes the proof. □

Theorem 2.4 (Haver's theorem; 7.1.6*). *Let X and Y be locally compact ANR's and let $f : X \to Y$ be a continuous surjection. Then the following statements are equivalent:*

(a) f is cell–like,

(b) f is a proper fine homotopy equivalence.

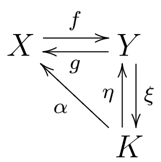
Proof. First we show (b) $\Longrightarrow$ (a). Since f is clearly proper (and so closed) surjection, we prove only that $f^{-1}(y)$ has trivial shape for every $y \in Y$. By lemma 2.1 it's enough to show that $f^{-1}(y)$ is contractible in one of its neighborhoods, say V. Since f is closed, we have a neighborhood U_0 of y in Y with $f^{-1}(U_0) \subseteq V$. Let $U_1 = Y \setminus \{y\}$. Then $\mathcal{U} = \{U_0, U_1\}$ is an open cover of Y, therefore there is a proper map $g : Y \to X$ such that, besides other things, $g \circ f : X \to X$ is proper $f^{-1}(\mathcal{U})$-homotopic to the identity, by homotopy say $H : X \times I \to X$. Now if $x \in f^{-1}(y)$, then $x \notin f^{-1}(U_1)$, so $H(\{x\} \times I) \subset f^{-1}(U_0) \subseteq V$. So $H \restriction V \times I : V \times I \to X$ is proper homotopy such that $H_0 = 1_V$ and $H_1(x) = g \circ f(x) = g(y)$ for $x \in f^{-1}(y)$. So this homotopy contracts $f^{-1}(y)$ in V to a point.

Now we prove (a) $\Longrightarrow$ (b). First let $\mathcal{V}$ be common open refinement of the following open covers of Y:

- $\mathcal{U}$,
- an open cover $\mathcal{V}_1$ of Y whose existence follows from proposition 1.3 ("in particular" part),
- an open cover $\mathcal{V}_2$ of Y whose existence again follows from proposition 1.3 (so any two $f^{-1}(\mathcal{V}_2)$-close maps into X are either both proper or both not).

Now let $\mathcal{U}_1$ be St^2–refinement of $\mathcal{V}$, $\mathcal{U}_2$ be St–refinement of $\mathcal{U}_1$ and $\mathcal{U}_3$ be a refinement of $\mathcal{U}_2$ which exists by theorem 1.9, i.e. any two $\mathcal{U}_3$-close maps are $\mathcal{U}_2$-homotopic. Now let K be a polytope which $\mathcal{U}_2$-dominates Y and let $\eta : K \to Y$ and $\xi : Y \to K$ be witnesses for this, i.e. $\eta \circ \xi$ is $\mathcal{U}_2$-homotopic to 1_Y.

By previous proposition 2.3 there exists a map $\alpha : K \to X$ such that $f \circ \alpha$ and η are $\mathcal{U}_3$-close. We put $g = \alpha \circ \xi$. The situation is:



Since for any $y \in Y$ we have $f \circ g(y) = f \circ \alpha(\xi(y))$ and $f \circ \alpha$ and η are $\mathcal{U}_3$-close, it follows that there exists $U \in \mathcal{U}_3$ containing both $f \circ \alpha(\xi(y))$ and $\eta(\xi(y))$. In other words, also $f \circ g$ and $\eta \circ \xi$ are $\mathcal{U}_3$-close, therefore $\mathcal{U}_2$-homotopic. So we have

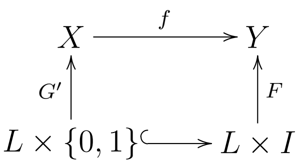
$$fg \sim_{\mathcal{U}_2} \eta\xi \sim_{\mathcal{U}_2} 1_Y.$$

Since $\mathcal{U}_2$ is star-refinement of $\mathcal{U}_1$, fg and 1_Y are $\mathcal{U}_1$-homotopic (so also $\mathcal{V}_1$- and $\mathcal{U}$-homotopic). Let $h : Y \times I \to Y$ be a witness for this fact, so it's a homotopy limited by $\mathcal{U}_1$ with $h_0 = 1_Y$ and $h_1 = fg$. Let us remark that since 1_Y is obviously proper and h is limited by $\mathcal{V}_1$, it follows that the whole homotopy h is proper.

Now we aim at showing that gf is $f^{-1}(\mathcal{U})$-homotopic to 1_X. Let $\mathcal{W}$ be a common open refinement of $f^{-1}(\mathcal{U}_1)$ and $(gf)^{-1}(f^{-1}(\mathcal{U}_1))$. Now let L be a polytope $\mathcal{W}$-dominating X by maps $\beta : X \to L$ and $\gamma : L \to X$. So $\gamma \circ \beta$ is $\mathcal{W}$-homotopic (so also $f^{-1}(\mathcal{U}_1)$-homotopic) to 1_X by a homotopy, say $H : X \times I \to X$. In particular, H is limited by $(gf)^{-1}(f^{-1}(\mathcal{U}_1))$, so $gfH : X \times I \to X$ is a homotopy joining $gf\gamma\beta$ with gf limited by $f^{-1}(\mathcal{U}_1)$. Summarizing, so far we have

$$gf \sim_{f^{-1}(\mathcal{U}_1)} gf\gamma\beta, \quad \gamma\beta \sim_{f^{-1}(\mathcal{U}_1)} 1_X. \tag{2.1}$$

Now define a map $F : L \times I \to Y$ by $F(x,t) = h(f\gamma(x), t)$. It's clearly continuous, $F_0 = f\gamma$ and $F_1 = fgf\gamma$. Next define $G' : L \times \{0,1\} \to X$ by expression $G'(x,0) = \gamma(x)$ and $G'(x,1) = gf\gamma(x)$. Again, it's clearly continuous and the diagram



commutes. Observe that $L \times I$ is a polytope and $L \times \{0,1\}$ is its subpolytope. Applying previous proposition 2.3 we get a continuous extension $G : L \times I \to X$ of G' such that $f \circ G$ and F are $\mathcal{U}_1$-close.

Define a continuous map $\Psi : X \times I \to X$ by $\Psi(x,t) = G(\beta(x),t)$. We have

$$\Psi(x,0) = G(\beta(x),0) = G'(\beta(x),0) = \gamma(\beta(x))$$

and

$$\Psi(x,1) = G(\beta(x),1) = G'(\beta(x),1) = gf\gamma(\beta(x)),$$

so Ψ is a homotopy on X joining $\gamma\beta$ and $gf\gamma\beta$.

We prove that Ψ is limited by $\mathrm{St}(f^{-1}(\mathcal{U}_1))$. It's enough to prove that $f \circ \Psi$ is limited by $\mathrm{St}(\mathcal{U}_1)$. Fix $x \in X$. For any $y \in Y$ there is some set in $\mathcal{U}_1$ containing $h(\{y\} \times I)$, so in particular $h(\{f\gamma\beta(x)\} \times I) \subset U$ for some $U \in \mathcal{U}_1$. But for any $t \in I$, $h(f\gamma\beta(x),t) = F(\beta(x),t)$ and since $f \circ G$ and F are $\mathcal{U}_1$-close, there is $U' \in \mathcal{U}_1$ containing both $F(\beta(x),t)$ and $f \circ G(\beta(x),t) = f \circ \Psi(x,t)$. So $f \circ \Psi(x,t) \in \mathrm{St}(U)$. Since $t \in I$ was arbitrary, we obtain $f \circ \Psi(\{x\} \times I) \subset \mathrm{St}(U)$. This is what we needed.

Putting $gf\gamma\beta \sim_{\mathrm{St}(f^{-1}(\mathcal{U}_1))} \gamma\beta$ together with (2.1) we have at last that $g \circ f$ and 1_X are $\mathrm{St}^2(f^{-1}(\mathcal{U}_1))$-homotopic, therefore they are also $f^{-1}(\mathcal{U})$- and $f^{-1}(\mathcal{V}_2)$-homotopic, by the definition of $\mathcal{U}_1$. It follows that the homotopy joining proper map 1_X and $g \circ f$ is whole proper.

It remains to show that also g is proper. But notice that for any compact $K \subset X$ we have $g^{-1}(K) = f(f^{-1}(g^{-1}(K))) = f((gf)^{-1}(K))$, which is compact by continuity of f and properness of gf. We are done. □

Haver's theorem has some corollaries.

Corollary 2.5 (7.1.7*). *Let X and Y be locally compact ANR's and let $f : X \to Y$ be cell-like. In addition, let $A \subset Y$ be compact. If U is a neighborhood of A in Y such that A is contractible in U then $f^{-1}(A)$ is contractible in $f^{-1}(U)$.*

Proof. Put $B = f^{-1}(A)$. Let $H : A \times I \to U$ be the homotopy which contracts A to a point. Denote $A' = H(A \times I)$. By compactness, A' is a closed subset of U. Take any open cover $\mathcal{U}$ of U consisting of subsets of U. By the theorem 2.4, there exists a proper map $g : Y \to X$ such that

(1) $f \circ g$ is proper $\mathcal{U}$-homotopic to 1_Y,

(2) $g \circ f$ is proper $f^{-1}(\mathcal{U})$-homotopic to 1_X.

We show that $g(A') \subset f^{-1}(U)$. Take any $a \in A'$. By (1) we have some $V \in \mathcal{U}$ with $\{a, fg(a)\} \subset V$. But then $g(a) \in f^{-1}(V) \subset f^{-1}(U)$. We are done.

By (2) there exists a $f^{-1}(\mathcal{U})$-homotopy $S : B \times I \to X$ such that $S_0 = 1_B$ and $S_1 = (g \circ f) \restriction B$. Clearly $S(B \times I) \subset f^{-1}(U)$.

Define $T : B \times I \to X$ as $T = g \circ H \circ (f \times 1_I)$. T is a homotopy connecting $(g \circ f) \restriction B$ with a constant function. Moreover $T(B \times I) \subset g \circ H(A \times I) = g(A') \subset f^{-1}(U)$.

We finish the proof by the observation that by "attaching" T to S we obtain required contraction of $f^{-1}(A)$ to a single point in $f^{-1}(U)$. □

Corollary 2.6 (7.1.8*,7.1.9*). *Let X and Y be locally compact ANR's and let $f : X \to Y$ be cell-like. If a compactum $K \subset Y$ has a trivial shape then $f^{-1}(K)$ has trivial shape. Therefore, any finite composition of cell-like maps between locally compact ANR's is also cell-like.*

Proof. Just apply the corollary 2.5 and the theorem 2.1. □

2.2 Z-Sets in ANR's

We modify a few statements from the book.

Theorem 2.7 (7.2.5*). *Let X be an ANR and let $A \subset X$ be closed. The following statements are equivalent:*

(a) $A \in \mathcal{Z}(X)$,

(d) for every open cover $\mathcal{U}$ of X there exists a continuous function $f : X \to X - A$ such that f and 1_X are $\mathcal{U}$-close. This f can be chosen to be proper.

The only change against the book is that f can be proper. But this is clear by proposition 1.3.

Lemma 2.8 (7.2.7*). *Let X be a locally compact ANR and let $A \in \mathcal{Z}_\sigma(X)$. If K is the locally compact space and K_0 is its closed subset, then for every open cover $\mathcal{U}$ of X and every proper function $f : K \to X$ there is a proper function $g : K \to X$ such that*

(1) f and g are $\mathcal{U}$-close,

(2) $f \restriction K_0 = g \restriction K_0$,

(3) $g(K - K_0) \cap A = \emptyset$.

Proof. Let $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is a Z-set in X and let $K - K_0 = \bigcup_{n=1}^{\infty} K_n$, where each K_n is compact (this is possible since $K - K_0$ is locally compact).

Let $\mathcal{A} = \{g \in \mathcal{C}(X, Y) \mid g \restriction K_0 = f \restriction K_0 \text{ and } g \text{ is proper}\}$. According to the proposition 1.4, it is Baire space. Now for $n, m \geq 1$ define

$$\mathcal{A}_{n,m} = \{g \in \mathcal{A} \mid g(K_n) \cap A_m = \emptyset\}.$$

We prove that these sets are open and dense in $\mathcal{A}$. Since K_n is compact, $g(K_n)$ is closed, also A_m is closed, therefore we easily see that $\mathcal{A}_{n,m}$ is open.

For proving that $\mathcal{A}_{n,m}$ is dense in $\mathcal{A}$, fix $g \in \mathcal{A}$ and an open cover $\mathcal{V}$ of of X. By proposition 1.3 it is clear that without loss of generality we can assume that all $\mathcal{V}$-close maps to g in $\mathcal{C}(K, X)$ are proper. Now let $\mathcal{V}_0$ be an open refinement of $\mathcal{V}$ such that any two $\mathcal{V}_0$-close maps to X are $\mathcal{V}$-homotopic (theorem 1.9). By theorem 2.7 there exists a proper function $\xi : X \to X - A_m$ which is $\mathcal{V}_0$-close to 1_X. But then also $\xi \circ g$ and g are $\mathcal{V}_0$-close and therefore they are $\mathcal{V}$-homotopic by homotopy, say $H : K \times I \to X$.

Let $\alpha : K \to I$ be an Urysohn function with $\alpha \restriction K_0 = 0$ and $\alpha \restriction K_n = 1$. Define $g' : K \to X$ by $g'(x) = H_{\alpha(x)}(x)$. We see that g' is $\mathcal{V}$-close to g (because the whole homotopy H is limited by $\mathcal{V}$) and for $x \in K_0$ we have $g'(x) = H_0(x) = g(x) = f(x)$. It follows that g' is proper and $g' \in \mathcal{A}$. Moreover $g'(K_n) = H_1(K_n) = \xi(g(K_n))$, so $g'(K_n)$ misses A_m. Consequently, we have $g' \in \mathcal{A}_{n,m}$, and since $g' \in N(g, \mathcal{V})$ we have at last that $\mathcal{A}_{n,m}$ is dense in $\mathcal{A}$.

It follows that

$$\mathcal{B} = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \mathcal{A}_{n,m}$$

is dense in $\mathcal{A}$. Now it suffices to take any map from $\mathcal{B}$ that is in the neighborhood $N(f, \mathcal{U})$ (such a map exists due to denseness of $\mathcal{B}$). □

Proposition 2.9 (7.2.10*). *Let X and Y be locally compact ANR's and let $f : X \to Y$ be cell-like. Then for each $A \in \mathcal{Z}(Y)$, each neighborhood V of $f^{-1}(A)$ and each open covering $\mathcal{U}$ of Y there is a proper homotopy $\alpha : X \times I \to X$ having the following properties:*

(1) α is limited by $f^{-1}(\mathcal{U})$,

(2) $\alpha_0 = 1_X$ and $\alpha_1(X) \cap f^{-1}(A) = \emptyset$,

(3) every α_t ($t \in I$) restricts to the identity on $X - V$.

Changes in the proof against the book. The open covers $\mathcal{E}$ and $\mathcal{F}$ will be not finite anymore, but we do not need this fact in the proof. We remark that for $\mathcal{F}$ to have the property marked in the book by (1) we need $\xi(Y)$ to be closed, but this is clear because we can choose ξ to be proper, and so closed (theorem 2.7). The rest of the proof does not need further changes.

To ensure the properness of α, notice that by proposition 1.3 we can without loss of generality assume that any $f^{-1}(\mathcal{U})$-close map to (the proper map) 1_X is also proper. Now from (1) follows properness of α. □

Proposition 2.10 (7.2.12*). *Let X and Y be locally compact ANR's and let $f : X \to Y$ be cell-like. If $A \subset Y$ is such that $f^{-1}(A) \in \mathcal{Z}(X)$ then $A \in \mathcal{Z}(Y)$.*

The proof given in the book works well also here, without any change. (Of course, we reference to generalized statements.)

2.3 The reformulation of the disjoint-cells property

We say that a map $f : X \to Y$ between spaces X and Y is *approximable* by maps having some property $\mathcal{P}$, if in every neighborhood of f in the space $\mathcal{C}(X, Y)$ (endowed with the topology described in the section 1.2) there is a map having property $\mathcal{P}$. In other words, for every open cover $\mathcal{U}$ of Y there is a map $g \in \mathcal{C}(X, Y)$ having property $\mathcal{P}$ such that f and g are $\mathcal{U}$-close.

We say that a space X has *disjoint-cells property* if for every $n \in \mathbb{N}$, every continuous function $f : I^n \times \{0, 1\} \to X$ is approximable by maps sending $I^n \times \{0\}$ and $I^n \times \{1\}$ to disjoint sets. We say that a space X has the *Z-approximation property*, if for every $n \in \mathbb{N}$, every continuous map $f : I^n \to X$ can be approximated by Z-maps.

Proposition 2.11 (7.3.2). *Let X be a topologically complete space. The following statements are equivalent:*

(a) X has the disjoint-cells property,

(b) $\mathcal{C}(Q, X)$ contains a countable dense set consisting entirely of Z-maps,

(c) $\mathcal{C}(Q, X)$ contains a countable dense set consisting entirely of Z-maps having pairwise disjoint images,

(d) every continuous function $f : Q \to X$ is approximable by Z-maps,

(e) X has the Z-approximation property.

The last proposition is just as in the book, the following two corollaries are generalized; though the proofs are the same as in the book, so we shall not give them here.

Corollary 2.12 (7.3.3*). *Let X be a locally compact ANR. Then the following statements are equivalent:*

(a) X has the disjoint-cells property,

(b) if K is any locally compact space then any proper map $f : K \to X$ can be approximated by proper maps $g, h : K \to X$ having the property that $g(K) \cap h(K) = \emptyset$.

Corollary 2.13 (7.3.4*). *Let X be a locally compact ANR with the disjoint-cells property. If K is the locally compact space and if $E, F, G \subseteq K$ are pairwise disjoint closed subsets of K then any proper map $f : K \to X$ such that $f \restriction G$ is a Z-map can be approximated by proper maps $g : K \to X$ such that*

(1) $g(E), g(F)$ and $g(G)$ are pairwise disjoint,

(2) $g \restriction G = f \restriction G$.

Now we add a few more equivalent statements to those in the theorem 7.3.5 (the main theorem in this section).

Theorem 2.14 (7.3.5*). *Let X be a locally compact* ANR*. The following statements are equivalent:*

- *(a) X has the disjoint-cells property,*
- *(h) for every locally compact space K, every proper map $f : K \to X$ is approximable by Z-embeddings,*
- *(i) for every locally compact space K and closed subset $K_0 \subset K$ and for every $Z \in \mathcal{Z}_\sigma(X)$, every proper map $f : K \to X$ such that $f \restriction K_0$ is a Z-embedding, is approximable by Z-embeddings $g : K \to X$ such that $f \restriction K_0 = g \restriction K_0$ and $g(K - K_0) \subset X - Z$.*

Proof. Let us remark that (h) and (i) are analogies to (e) and (g) from the theorem.

Implications (i) $\implies$ (h) $\implies$ (e) are trivialities.

We shall prove (a) $\implies$ (i). By proposition 2.11 there exists a countable dense subset $F \subset \mathcal{C}(Q, X)$ consisting entirely of Z-maps. Put

$$\mathcal{A} = \{g \in \mathcal{C}(X, Y) \mid g \restriction K_0 = f \restriction K_0 \text{ and } g \text{ is proper}\}.$$

This is Baire space by proposition 1.4.

Let $\mathcal{B}$ be an open countable basis for the topology of $K - K_0$ such that for every $B \in \mathcal{B}$ the set $\overline{B}$ is compact (this is possible since $K - K_0$ is locally compact and second countable). For each pair $A, B \in \mathcal{B}$ with $\overline{A} \cap \overline{B} = \emptyset$ define

$$\mathcal{E}_{A,B} = \{g \in \mathcal{A} \mid g(\overline{A}), g(\overline{B}) \text{ and } g(K_0) \text{ are pairwise disjoint}\}. \tag{2.2}$$

By corollary 2.13 each $\mathcal{E}_{A,B}$ is dense in $\mathcal{A}$ and it is clearly open.

Write $Z = \bigcup_{n\in\mathbb{N}} Z_n$, where each Z_n is a Z-set in X. For $B \in \mathcal{B}$, $h \in F$ and $n \in \mathbb{N}$ let

$$\mathcal{F}_{B,h,n} = \left\{g \in \mathcal{A} \mid g(\overline{B}) \cap (h(Q) \cup Z_n) = \emptyset\right\}. \tag{2.3}$$

Since $\overline{B} \subset K - K_0$ and $h(Q) \cup Z_n$ is a Z-set, lemma 2.8 implies that each $\mathcal{F}_{B,h,n}$ is dense in $\mathcal{A}$. It is also open in $\mathcal{A}$, because $g(\overline{B})$ and $h(Q) \cup Z_n$ are both closed in X.

Now, it is clear that the collection of all $\mathcal{F}_{B,h,n}$'s, as well as the collection of all $\mathcal{E}_{A,B}$'s, is clearly countable. It follows that

$$\mathcal{G} = \bigcap_{A,B\in\mathcal{B}} \mathcal{E}_{A,B} \cap \bigcap_{B\in\mathcal{B},h\in F,n\in\mathbb{N}} \mathcal{F}_{B,h,n}$$

is also dense in $\mathcal{A}$. By (2.2) we have that g is an embedding. By (2.3) it is clear that for any $g \in \mathcal{G}$ we have $g(K - K_0) \subset X - Z$. Furthermore, for each such $g \in \mathcal{G}$ we have $g(K - K_0) \cap \bigcup_{h\in F} h(Q) = \emptyset$, so by denseness of F, $g(K) = g(K - K_0) \cup f(K_0)$ is closed and satisfies the definition of Z-set. Therefore g is Z-embedding. This finished the proof, since we see that any map $g \in \mathcal{G}$ close to f has required properties. □

Theorem 2.15 (7.3.6). *(1) If a space X has the disjoint-cells property then for any space Y, $X \times Y$ has the disjoint-cells property.*

(2) Every $\mathcal{Q}$-manifold is an ANR *and has the disjoint-cells property.*

2.4 Z-sets in $\mathcal{Q}$-manifolds

Corollary 2.16 (7.4.4). *(1) If M and N are $\mathcal{Q}$-manifolds and $M \subseteq N$ is a Z-set, then M is collared in N.*

(2) If M is $\mathcal{Q}$-manifold then $M \times [0, 1)$ is homeomorphic to an open subset of $\mathcal{Q}$.

Theorem 2.17 (7.4.5*). *Let M be a $\mathcal{Q}$-manifold. If $A \in \mathcal{Z}(M)$ then there is a neighborhood of A which is homeomorphic to an open subset of $\mathcal{Q}$.*

The proof of this theorem is identical with that one in the book.

Lemma 2.18 (7.4.6). *Let $A, B \in \mathcal{Z}(\mathcal{Q})$. Then for every neighborhood V of B and for every open cover $\mathcal{U}$ of V there is an isotopy $H : \mathcal{Q} \times I \to \mathcal{Q}$ such that*

- *(1) $H_0 = 1_{\mathcal{Q}}$ and $H_1(B) \cap A = \emptyset$,*
- *(2) $H_t \restriction \mathcal{Q} - V = 1_{\mathcal{Q}-V}$ for every $t \in I$,*
- *(3) $H \restriction V \times I \to V$ is limited by $\mathcal{U}$.*

Now we aim at the Z-set unknotting theorem for locally compact spaces. First we need one definition and series of lemmas. If X is a space and $r : X \to I$ is a continuous map, then we define *the variable product*

$$X \times_r I = \bigcup \{\{x\} \times [0, r(x)] \mid x \in X\} \subset X \times I.$$

Lemma 2.19. *Let $V \subseteq \mathcal{Q}$ be open, A a locally compact space, $\mathcal{U}$ an open cover of V and $F : A \times I \to V$ be Z-embedding and let $r, s : A \to [\frac{1}{2}, 1]$ be continuous maps such that the set*

$$D = \{a \in A \mid r(a) \neq s(a)\}$$

has compact closure. Then there exists an isotopy $H : \mathcal{Q} \times I \to \mathcal{Q}$ such that $H_0 = 1_{\mathcal{Q}}$ and H_1 takes $F(A \times_r I)$ to $F(A \times_s I)$ by the formula

$$H_1(F(a, t)) = F(a, t \cdot \tfrac{s(a)}{r(a)})$$

for all $(a, t) \in A \times_r I$. Moreover, if F is limited by $\mathcal{U}$, then we may construct H in such a way that $H_t \restriction \mathcal{Q} - V = 1_{\mathcal{Q}-V}$ for all $t \in I$ and $H \restriction V \times I \to V$ is limited by $\mathcal{U}$.

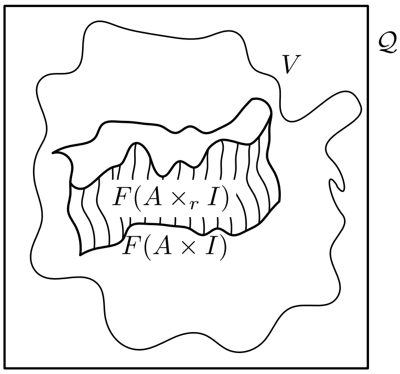


Figure 2.1: The variable product

Proof. Denote $\vec{\mathcal{Q}} = \prod_{i=2}^{\infty} J$ and regard $\mathcal{Q} = \vec{\mathcal{Q}} \times [-1, 2]$. Put A in some endface of $\vec{\mathcal{Q}}$, so $A \in \mathcal{Z}(\vec{\mathcal{Q}})$. Consequently, $A \times I \in \mathcal{Z}(\mathcal{Q})$. By the extension theorem 1.12 there exists a homeomorphism $f : \mathcal{Q} \to \mathcal{Q}$ extending $F^{-1} : F(A \times I) \to A \times I$.

The idea for the rest of the proof is simple: first we take $F(A \times I)$ to $A \times I \subset \vec{\mathcal{Q}} \times [-1, 2]$ by f. Now we perform a "simple push in [-1,2]–coordinate" which takes $A \times_r I$ to $A \times_s I$. We realize this by an isotopy $G : \mathcal{Q} \times I \to \mathcal{Q}$ which is identity everywhere except on possibly $N \times [-p, 1 + p]$ (where $N \subset A$ is any compact neighborhood of D and choice of p is explained later on). After this, we take the situation back to original position by f^{-1}.

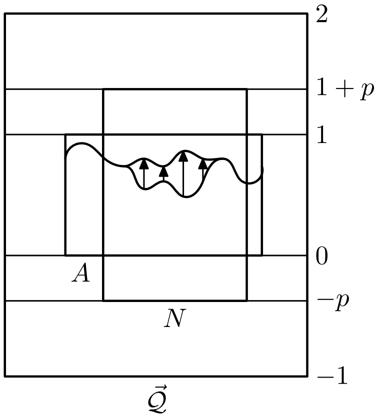


Figure 2.2: The isotopy G

We give the details of the construction of G. Assume that F is limited by $\mathcal{U}$ and assume that $\mathcal{U}$ consists of subsets of V. Recall that N is compact neighborhood of D. It follows from these facts that there exists $p \in (0, 1)$ such that

$$\text{for any } x \in N : \quad \{x\} \times [-p, 1 + p] \subset f(U) \text{ for some } U \in \mathcal{U}. \tag{$*$}$$

Let $G : \mathcal{Q} \times I \to \mathcal{Q}$ be defined by

$$G_y(x,t) = (x, ty\tfrac{s(x)}{r(x)} + t(1-y)) \quad \text{for } (x,t) \in N \times [0, r(x)]$$
$$G_y(x,t) = (x, t + y(s(x) - t + \tfrac{1+p-s(x)}{1+p-r(x)}(t - r(x)))) \quad \text{for } (x,t) \in N \times [r(x), 1+p]$$
$$G_y(x) = x \quad \text{for } x \in \mathcal{Q} - (N \times [0, 1+p])$$

A straightforward check shows that G is indeed an isotopy which has all the required properties. Now $H = f^{-1} \circ G \circ f$ is the one we seek. □

Remark. It follows from the construction of G that H moves points only "along the segments" $\{x\} \times [-p, 1+p]$ for $x \in N$. This fact, together with the condition $(*)$, shows that not only that H is limited by $\mathcal{U}$, but

$$F(\{x\} \times I) \subset U \in \mathcal{U} \quad \Longrightarrow \quad H(F(\{x\} \times I) \times I) \subset U.$$

We shall make use of this fact later on.

Lemma 2.20. *Let $V \subseteq \mathcal{Q}$ be open, A a locally compact space, $\mathcal{U}$ be an open cover of V and $F : A \times I \to V$ be a proper map such that F_0 and F_1 are Z-embeddings and $F_0(A) \cap F_1(A) = \emptyset$. Then there exists an isotopy $H : \mathcal{Q} \times I \to \mathcal{Q}$ such that $H_0 = 1_{\mathcal{Q}}$, $H_1 F_0 = F_1$ and $H_t \restriction \mathcal{Q} - V = 1_{\mathcal{Q}-V}$ for all $t \in I$. Moreover, if F is limited by $\mathcal{U}$, then we may construct H in such a way that $H \restriction V \times I \to V$ is limited by $\mathcal{U}$.*

Proof. By theorem 2.14 we can approximate F by Z-embeddings $\bar{F}$ such that $\bar{F}_0 = F_0$ and $\bar{F}_1 = F_1$. So without loss of generality we may assume that F is a Z-embedding. Furthermore, assume that $\mathcal{U}$ consists of subsets of V and that F is limited by $\mathcal{U}$.

We construct an isotopy H which has all the required properties except that $H_1 F_1 = F_0$ instead of $H_1 F_0 = F_1$. But it is clear that this suffices. The idea of the proof is that we make use of the previous lemma to move F_1 to F_0 "piece by piece". First we push F_1 to $F_{1/2}$ and then apply the same construction to move $F_{1/2}$ to F_1.

We write $A = \bigcup_{n \in \mathbb{N}} A_n$, where all A_n's are compact and $A_n \subset \operatorname{int} A_{n+1}$ for $n \in \mathbb{N}$. Construct a sequence of continuous functions $r_n : A \to [\frac{1}{2}, 1]$ with the following properties (for all $n \in \mathbb{N}$):

- $r_n(A_{4n} - \operatorname{int} A_{4n-1}) \subset \{\frac{1}{2}\}$,
- $r_n((A - \operatorname{int} A_{4n+1}) \cup A_{4i-2}) \subset \{1\}$.

Such functions are just a little bit modified Urysohn functions for appropriate sets. In addition, define $r_0 : A \to I$ by $r_0 \equiv 1$. The previous lemma gives us a sequence of isotopies $G^n : \mathcal{Q} \times I \to \mathcal{Q}$ such that the following conditions hold (for each $n \in \mathbb{N}$):

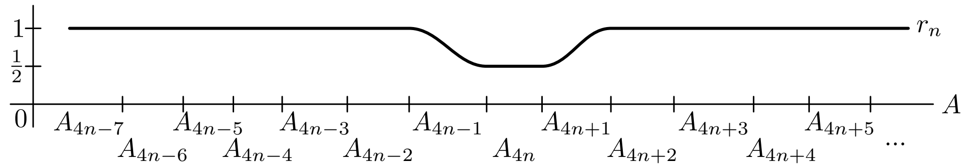


Figure 2.3: The graph of r_n

- G^n is limited by $\mathcal{U}$ (in a sense of the remark after 2.19),
- G^n_t is an identity on $\mathcal{Q} - V$ for every $t \in I$,
- $G^n_0 = 1_{\mathcal{Q}}$ and G^n_1 takes $F(A \times_{r_0} I)$ to $F(A \times_{r_n} I)$.

Now let

$$G^{(1)} = \lim_{n\to\infty} G^n \circ G^{n-1} \circ \cdots \circ G^1.$$

It is clear from the construction of the G^n's that each point of $\mathcal{Q}$ is moved only by at most one of them, so the limit is well-defined. Moreover, for every $t \in I$, $G^{(1)}_t$ is an identity on $\mathcal{Q} - V$ and $G^{(1)}_1$ takes $F(A \times_1 I)$ to $F(A \times_r I)$, where $r(x) = 1 + \sum_{i=1}^{\infty}(r_n(x) - 1)$. Notice that for every $x \in A$, at most one of the r_n's is not equal to 1, so the sum is well-defined. Furthermore, $G^{(1)}$ is limited by $\mathcal{U}$ in a sense of the remark after the lemma 2.19.

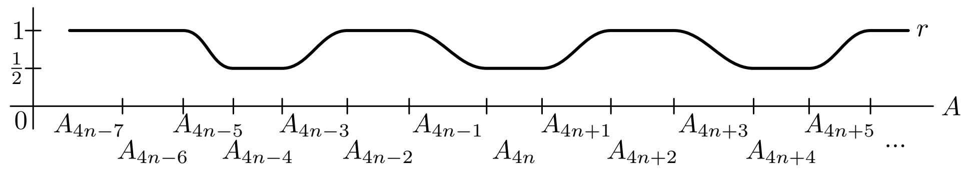


Figure 2.4: The graph of r

We put

$$s_1(x) = \begin{cases} r(x) & \text{for } x \in A - A_3, \\ \frac{1}{2} & \text{for } x \in \operatorname{int} A_4, \end{cases}$$

and for $n \geq 2$ we define

$$s_n(x) = \begin{cases} r(x) & \text{for } x \in \operatorname{int} A_{4n-4} \cup (A - A_{4n-1}), \\ \frac{1}{2} & \text{for } x \in \operatorname{int} A_{4n} - A_{4n-6}. \end{cases}$$

It is easy to see that each $s_n : A \to [\frac{1}{2}, 1]$ is continuous, since both cases in the definition are on the open sets and they agree on the intersection. In addition, define $s_0 : A \to I$ by $s_0 \equiv \frac{1}{2}$. We again use the previous lemma to get a sequence of isotopies $'G^n : \mathcal{Q} \times I \to \mathcal{Q}$ with the properties (for every $n \in \mathbb{N}$):

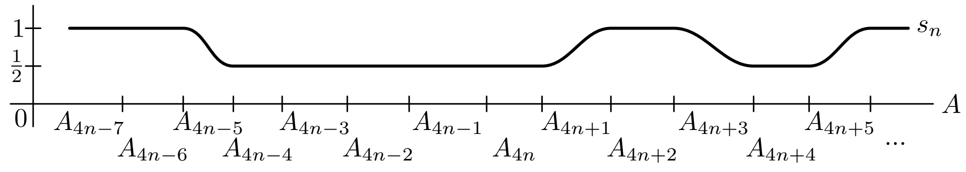


Figure 2.5: The graph of s_n

- $'G^n$ is limited by $\mathcal{U}$ (in a sense of the remark after 2.19),
- $'G^n_t$ is an identity on $\mathcal{Q} - V$ for every $t \in I$,
- $'G^n_0 = 1_{\mathcal{Q}}$ and $'G^n_1$ takes $F(A \times_{s_n} I)$ to $F(A \times_{s_0} I)$.

We put

$$G^{(2)} = \lim_{n\to\infty} {'G^n} \circ \cdots \circ {'G^1}.$$

Again, $G^{(2)}$ has all the properties as $G^{(1)}$ except that $G^{(2)}_1$ takes $F(A \times_r I)$ to $F(A \times_{1/2} I)$.

We define an isotopy $G : \mathcal{Q} \times I \to \mathcal{Q}$ by

$$G_t = \begin{cases} G^{(1)}_{2t} & \text{for } t \in [0, \frac{1}{2}] \\ G^{(2)}_{2t-1} \circ G^{(1)}_1 & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

It clearly has the properties as $G^{(1)}$, except that G_1 takes $F(A\times_1 I)$ to $F(A\times_{1/2})$. Consequently, $G_1F_1 = F_{1/2}$. We can use the fact in the remark following the lemma 2.19 to see that G is indeed limited by $\mathcal{U}$ in a sense of this remark.

By the analogous construction we obtain an isotopy $K : \mathcal{Q} \times I \to \mathcal{Q}$ which fulfills:

- for every $t \in I$, K_t is identity on $\mathcal{Q} - V$,
- is limited by $\mathcal{U}$ (in a sense of the remark after 2.19),
- $K_0 = 1_{\mathcal{Q}}$ and $K_1F_{1/2} = F_0$.

Now just put

$$H_t = \begin{cases} G_{2t} & \text{for } t \in [0, \frac{1}{2}] \\ K_{2t-1} \circ G_1 & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$

It is clear that H has all the required properties. □

Lemma 2.21 (7.4.8*). *Let $V \subseteq \mathcal{Q}$ be open, A a locally compact space, $\mathcal{U}$ be an open cover of V and $F : A \times I \to V$ be a proper map such that F_0 and F_1 are Z-embeddings. Then there exists an isotopy $H : \mathcal{Q} \times I \to \mathcal{Q}$ such that $H_0 = 1_{\mathcal{Q}}$, $H_1F_0 = F_1$ and $H_t \restriction \mathcal{Q} - V = 1_{\mathcal{Q}-V}$ for all $t \in I$. Moreover, if F is limited by $\mathcal{U}$, then we may construct H in such a way that $H \restriction V \times I \to V$ is limited by $\mathcal{U}$.*

Proof. The idea behind this proof is that first we move $F_0(A)$ away from $F_1(A)$ by sufficiently "small" isotopy (lemma 2.18), and then apply previous lemma.

Assume that $\mathcal{U}$ consists of subsets of V and that F is limited by $\mathcal{U}$. We claim that there is an open cover $\mathcal{V}$ of $F_0(A)$ with the property

(1) for all $V \in \mathcal{V}$ and $x \in A$ with $F_0(x) \in V$ there exists $U \in \mathcal{U}$ such that $V \cup F(\{x\} \times I) \subset U$.

To this end, take $x \in A$. Since F is limited by $\mathcal{U}$, there is $U_x \in \mathcal{U}$ with $F(\{x\} \times I) \subset U_x$. Consequently, $F^{-1}(U_x)$ is an open neighborhood of $\{x\} \times I$ in $A \times I$. By compactness of I, there exists an open neighborhood W_x of x in A such that $W_x \times I \subset F^{-1}(U_x)$. Since F_0 is an embedding, $F_0(W_x)$ is an open neighborhood of $F_0(x)$ in $F_0(A)$. Consequently, there is an open set V_x in M such that $V_x \cap F_0(A) \subset F_0(W_x) \cap U_x$. Put $\mathcal{V} = \{V_x \mid x \in X\}$. To verify the condition (1), notice that if $F_0(y) \in V_x$ for some $y \in A$, we have that $y \in W_x$ and consequently $F(\{y\} \times I) \subset U_x$. Since $V_x \subset U_x$, the argument is finished.

Denote $V' = \bigcup \mathcal{V}$. Notice that $V' \subset V$. By lemma 2.18 there is an isotopy $S : \mathcal{Q} \times I \to \mathcal{Q}$ such that

- $S_0 = 1_{\mathcal{Q}}$ and $S_1(F_0(A)) \cap F_1(A) = \emptyset$,
- $S_t \restriction \mathcal{Q} - V' = 1_{\mathcal{Q}-V'}$ for every $t \in I$,
- $S \restriction V' \times I \to V'$ is limited by $\mathcal{V}$.

Define $\tilde{F} : A \times I \to \mathcal{Q}$ by

$$\tilde{F}(x,t) = \begin{cases} S_{1-2t}(F_0(x)) & \text{for } t \in [0, \frac{1}{2}], \\ F_{2t-1}(x) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

In fact, $\tilde{F}$ is just S "backwards" followed by F. Using (1) it is easy to see that $\tilde{F}$ is limited by $\mathcal{U}$. More precisely, for $x \in A$ there is $V \in \mathcal{V}$ containing $S(F_0(x) \times I)$, in particular containing also $S_0(F_0(x)) = F_0(x)$. Consequently, there exists $U \in \mathcal{U}$ such that $F(\{x\} \times I) \cup V \subset U$. But then also $H(\{x\} \times I) \subset U$.

It is clear that $\tilde{F}_0(A) \cap \tilde{F}_1(A) = \emptyset$. Therefore, by lemma 2.20 there exists an isotopy $T : \mathcal{Q} \times I \to \mathcal{Q}$ such that

- $T_0 = 1_{\mathcal{Q}}$ and $T_1 \tilde{F}_0 = \tilde{F}_1$,
- $T_t \restriction \mathcal{Q} - V = 1_{\mathcal{Q}-V}$ for every $t \in I$,
- $T \restriction V \times I \to V$ is limited by $\mathcal{U}$ (in a sense of the remark after 2.19).

Now define $H : \mathcal{Q} \times I \to \mathcal{Q}$ by

$$H(x,t) = \begin{cases} S_{2t}(x) & \text{for } t \in [0, \frac{1}{2}], \\ T_{2t-1}(S_1(x)) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

We see that $H_0 = 1_{\mathcal{Q}}$ and

$$H_1 F_0 = T_1 S_1 F_0 = T_1 \tilde{F}_0 = \tilde{F}_1 = F_1.$$

Since both S and T are identities on $\mathcal{Q} - V$, we have that $H_t \upharpoonright \mathcal{Q} - V = 1_{\mathcal{Q}-V}$.

It remains to show that $H \upharpoonright V \times I$ is limited by $\mathcal{U}$, but this is easily done by similar argument as for $\tilde{F}$ (using (1) and the remark after the lemma 2.19). □

Theorem 2.22 (The Z-set unknotting theorem; 7.4.9*). *Let M be a $\mathcal{Q}$-manifold, $\mathcal{U}$ be an open cover of M and $V \subset M$ be open. If A is the locally compact space and if $F : A \times I \to V$ is a proper homotopy limited by $\mathcal{U}$ such that F_0 and F_1 are Z-embeddings, then there exists an isotopy $H : M \times I \to M$ limited by $\mathcal{U}$ such that $H_0 = 1_M$, $H_1 F_0 = F_1$ and $H_t \upharpoonright M - V = 1_{M-V}$ for every $t \in I$.*

Proof. The strategy is that we reduce the situation in a $\mathcal{Q}$-manifold to the situation in the Hilbert Cube using theorem 2.17.

On $A \times I$ we define an equivalence relation $\sim$ by the statement

$$(x, 0) \sim (y, 1) \text{ iff } F_0(x) = F_1(y).$$

It is clear that we have well-defined continuous map $F/\!\sim\, : (A \times I)/\!\sim\, \to M$ which is Z-embedding on $A \times \{0, 1\}/\!\sim$. By the theorems 2.15 and 2.14 we can now approximate $F/\!\sim$ by Z-embeddings from $(A \times I)/\!\sim$, and it's clear that we can use them to construct Z-maps $G : A \times I \to M$ approximating F with $G_0 = F_0$ and $G_1 = F_1$.

So we may assume without loss of generality that $F(A \times I)$ is Z-set in M. By theorem 2.17 there is a neighborhood U_1 of $F(A \times I)$ which is homeomorphic to an open subset of $\mathcal{Q}$. By abuse of notation we assume that U_1 *is* an open subset of the Hilbert cube. Then $W_1 = U_1 \cap V$ is an open set (in U_1) containing $F(A \times I)$. Consequently, there exists an open neighborhood W of the closed set $F(A \times I)$ such that $F(A \times I) \subset W \subset \overline{W} \subset W_1$. (This is easy, just take an Urysohn function $u : M \to I$ such that $u \upharpoonright F(A \times I) \equiv 1$ and $u \upharpoonright W_1 \equiv 0$; then put $W = u^{-1}((\frac{1}{2}, 1])$.) Using theorem 2.7 twice we have that $F(A \times I) \in \mathcal{Z}(Q)$. We denote $\mathcal{V} = \{U \cap W \mid U \in \mathcal{U}\}$. By lemma 2.21 there exists an isotopy $G : \mathcal{Q} \times I \to \mathcal{Q}$ such that $G_0 = 1_{\mathcal{Q}}$, $G_1 F_0 = G_1$, $G_t \upharpoonright \mathcal{Q} - W = 1_{\mathcal{Q}-W}$ and $G : W \times I \to W$ is limited by $\mathcal{V}$. It follows that the mapping $H : M \times I \to M$ defined by

$$H(x, t) = \begin{cases} G(x, t) & \text{for } x \in W, \\ x & \text{for } x \in M - W \end{cases}$$

is an isotopy limited by $\mathcal{U}$ which fulfills our requirements. □

2.5 The proof Toruńczyk's Theorem

Lemma 2.23 (7.5.1*). *Let X be a locally compact space such that $X \times \mathcal{Q} \approx X$. Then the projection $\pi : X \times J \to X$ is a near homeomorphism.*

The proof of this lemma is exactly the same as in the book.

Now we prove some statements about the mapping cylinders. We use the same notation as introduced in the section 1.2.

Lemma 2.24 (7.5.2*). *Let M be a $\mathcal{Q}$-manifold, let X be a locally compact* ANR *and let $f : M \to X$ be cell-like. Then $\pi_f : M \times I \to M(f)$ is a near homeomorphism.*

Proof. We make use of Bing Shrinking Criterion 1.13. Since π_f is proper surjection, it's enough to prove that it's shrinkable. So, fix an open cover $\mathcal{U}$ of $M \times I$ and an open cover $\mathcal{V}$ of $M(f)$. We produce a homeomorphism $h : M \times I \to M \times I$ such that

(1) π_f and $\pi_f \circ h$ are $\mathcal{V}$-close,

(2) for every $x \in X$ there exists a $U \in \mathcal{U}$ with $h(f^{-1}(x) \times \{1\}) \subset U$.

Since $\pi_f \restriction M \times [0, 1)$ is one-to-one, h will be desired shrinking homeomorphism. For convenience, we identify M with $M \times \{1\}$.

Let $\mathcal{V}_1$ be open star-refinement of $\mathcal{V}$ and let $\mathcal{V}_2 = \{V \cap X \mid V \in \mathcal{V}_1\}$. It is an open cover of X. By the theorem 2.15 M is the locally compact ANR with the disjoint-cells property. Since $f : M \to X$ is cell-like, by the theorem 2.4 there exists a proper map $g : X \to M$ such that

(3) $g \circ f$ is proper $f^{-1}(\mathcal{V}_2)$-homotopic to 1_M,

(4) $f \circ g$ is proper $\mathcal{V}_2$-homotopic to 1_X.

By the theorem 1.9 there is an open refinement $\mathcal{U}_1$ of $f^{-1}(\mathcal{V}_2)$ such that any two $\mathcal{U}_1$-close maps into M are $f^{-1}(\mathcal{V}_2)$-homotopic. Moreover, let $\mathcal{U}_2$ be the open cover whose existence follows from the proposition 1.3, so any two $\mathcal{U}_2$-close maps into M are both proper or both not. Finally, let $\mathcal{U}_3$ be common open refinement of $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathrm{St}(\mathcal{U})$.

By the theorem 2.14, a proper map $g \circ f : M \to M$ is approximable by Z-embeddings. So there exists a continuous map $\xi : M \to M$ which is $\mathcal{U}_3$-close to $g \circ f$. It follows that it has the following properties:

(5) ξ is Z-embedding,

(6) ξ is proper $f^{-1}(\mathcal{V}_2)$-homotopic to $g \circ f$,

(7) for every $x \in X$ there is some $U \in \mathcal{U}$ such that $\xi f^{-1}(x) \subset U$.

Observe that (7) is true since $(g \circ f)(f^{-1}(x))$ is one-point set for every $x \in X$ and $g \circ f$ and ξ are $\mathrm{St}(\mathcal{U})$-close.

By (3) and (6) we have

$$\xi \sim_{f^{-1}(\mathcal{V}_2)} g \circ f \sim_{f^{-1}(\mathcal{V}_2)} 1_M.$$

Since $\pi_f \restriction M = f$ and $\mathcal{V}_1$ is star-refinement of $\mathcal{V}$, it follows that $\xi \sim_{\pi_f^{-1}(\mathcal{V})} 1_M$ by a proper homotopy. Notice that also $1_M : M \to M \subset M \times I$ is a Z-embedding. Now use the unknotting theorem 2.22 to produce a homeomorphism $h : M \times I \to M \times I$ such that

(8) $h \circ 1_M = \xi$ (i.e. $h \restriction M = \xi$),

(9) h and $1_{M\times I}$ are $\pi_f^{-1}(\mathcal{V})$-close.

(It is the 1-level of the isotopy that exists by the theorem.) It remains to check that h is the required shrinking homeomorphism. But by (9) it's clear that (1) holds. Moreover, by (8) h extends ξ and finally (7) ensures the property (2). We are done. □

Corollary 2.25 (7.5.3*). *Let M be a $\mathcal{Q}$-manifold, let X be a locally compact ANR and let $f : M \to X$ be cell-like. Then $M(f)$ is a $\mathcal{Q}$-manifold.*

Proposition 2.26 (7.5.4*). *Let M be a $\mathcal{Q}$-manifold, let X be a locally compact ANR and let $f : M \to X$ be cell-like. Then for every $Y \in \mathcal{Z}(X)$ and an open cover $\mathcal{F}$ of X there is a near homeomorphism $g : M(f) \to M(f)$ such that*

(1) $c(f) \circ g$ and $c(f)$ are $\mathcal{F}$-close,

(2) $g \restriction g^{-1}(Y) = c(f) \restriction c(f)^{-1}(Y)$.

Proof. We recall that we think of M and X as being the subspaces of $M(f)$. Let Z be the space we obtain from $M(f)$ by identifying each set of the form $c(f)^{-1}(y)$, $y \in Y$, to a single point (Z is something like reduced mapping cylinder; see figure 2.6). Let $q : M(f) \to Z$ be a quotient map and let $q_1 : Z \to X$ be such continuous surjective map that the following diagram commutes:

$$\begin{array}{ccc} M(f) & \xrightarrow{q} & Z \\ & \searrow^{c(f)} & \downarrow q_1 \\ & & X. \end{array}$$

Notice that q is obviously surjective and since for any compact $A \subset Z$, $q^{-1}(A)$ is closed subset of compact set $c(f)^{-1}(q_1(A))$, q is also proper.

The next step of the proof we formulate as a lemma.

Lemma 2.27. *We assume the situation as in the proof. For every open cover $\mathcal{U}$ of Z and $\mathcal{V}$ of $M(f)$ we shall construct a homeomorphism $h : M(f) \to M(f)$ such that*

(3) $q \circ h$ and q are $\mathcal{U}$-close,

(4) $h \restriction X = 1_X$,

(5) for every $y \in Y$ there is $V \in \mathcal{V}$ with $h(c(f)^{-1}(y)) \subset V$.

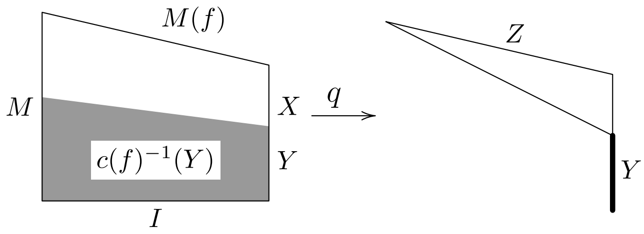


Figure 2.6: Reduced mapping cylinder

Since the fibers of q are exactly the sets $c(f)^{-1}(y)$ for $y \in Y$, (3) and (5), according to Bing Shrinking Criterion 1.13, give us the fact that q is a near homeomorphism. We shall use the fact (4) later on.

Proof of the lemma. First we shall construct a homeomorphism $h_1 : M(f) \to M(f)$ which takes $c(f)^{-1}(Y)$ away from M. For every $y \in Y$ choose its open neighborhood W_y in X such that $c(f)^{-1}(W_y) \subseteq q^{-1}(U)$ for some $U \in \mathcal{U}$. This can be easily done: choose $U \in \mathcal{U}$ such that $y \in U$; $q^{-1}(U)$ is an open neighborhood of the set $q^{-1}(y) = c(f)^{-1}(y)$; now existence of such W_y follows from closedness of the mapping $c(f)$.

Put $\mathcal{W} = \{W_y \mid y \in Y\} \cup \{X - Y\}$. This is an open cover of X. Let $\mathcal{W}'$ be its open refinement such that

- $\mathcal{W}' \overset{*}{<} \mathcal{W}$ and $\mathcal{W}' \overset{*}{<} \mathcal{V}$,
- $\mathcal{W}'$ is star-finite,
- every $W \in \mathcal{W}'$ has compact closure.

By the proposition 2.9 there exists a proper homotopy $\alpha : M \times I \to M$ such that

(6) α is limited by $f^{-1}(\mathcal{W}')$ $(= \{c(f)^{-1}(W) \cap M \mid W \in \mathcal{W}'\})$,

(7) $\alpha_1(M) \cap f^{-1}(Y) = \emptyset$ $(= \alpha_1(M) \cap c(f)^{-1}(Y))$,

(8) $\alpha_0 = 1_M$.

By the theorem 2.15, M is the locally compact ANR, by the theorem 2.14 we can approximate α_1 by embeddings, and by the theorem 1.9 "close" maps into ANR's are homotopic by "small" homotopy. Moreover, by the proposition 1.3, maps "close" to proper are also proper. Putting these facts together, we may assume that α_1 is an embedding.

The sets $M \times \{0\}$ and $M \times \{\frac{1}{2}\}$ are clearly Z-sets in the $\mathcal{Q}$-manifold $M \times [0, \frac{1}{2}]$. By the theorem 2.22 there exists a homeomorphism $\bar{h}_1 : M \times [0, \frac{1}{2}] \to M \times [0, \frac{1}{2}]$ such that (it's indeed the 1-level of the isotopy mentioned by the theorem):

(9) $\bar{h}_1 \restriction M \times \{\frac{1}{2}\} = 1_{M \times \{\frac{1}{2}\}}$,

(10) $\bar{h}_1(\alpha_1(M) \times \{0\}) = M \times \{0\}$,

(11) $\bar{h}_1$ and $1_{M\times[0,\frac{1}{2}]}$ are $\mathcal{E}$-homotopic, where

$$\mathcal{E} = \{c(f)^{-1}(W) \cap (M \times [0, \tfrac{1}{2}]) \mid W \in \mathcal{W}'\}.$$

Finally, we let $h_1 : M(f) \to M(f)$ be the union of $\bar{h}_1 : M \times [0, \frac{1}{2}] \to M \times [0, \frac{1}{2}]$ and the identity on the rest $1_{M(f)-M\times[0,\frac{1}{2})}$. Since they agree on the intersection $M \times \{\frac{1}{2}\}$, it's clear that h_1 is a homeomorphism.

So far we have constructed a homeomorphism $h_1 : M(f) \to M(f)$ with the following properties:

(12) h_1 and $1_{M(f)}$ are $c(f)^{-1}(\mathcal{W}')$-close,

(13) $h_1 \restriction (M(f) - M \times [0, \frac{1}{2}]) = 1_{M(f)-M\times[0,\frac{1}{2}]}$,

(14) $h_1(c(f)^{-1}(Y)) \cap M = \emptyset$.

Remark that (14) follows from (7) and (10); the rest of the conditions are clear.

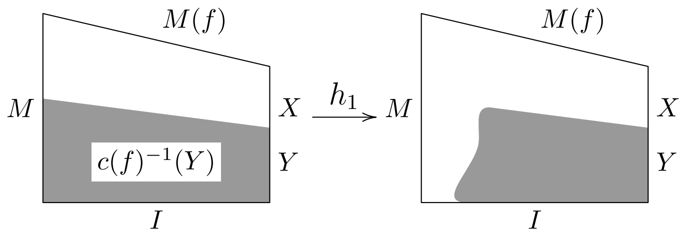


Figure 2.7: The mapping h_1.

Note that the set $O = \mathrm{St}(Y, \mathcal{W}')$ is the open neighborhood of Y in X. We now establish the fact

(15) for every $y \in Y$ we have $c(f)^{-1}(y) \cup h_1(c(f)^{-1}(y)) \subset c(f)^{-1}(\mathrm{St}(y, \mathcal{W}')) \subset c(f)^{-1}(O)$.

Take an arbitrary $y \in Y$ and pick $z \in c(f)^{-1}(y)$. By (12) there is $V \in \mathcal{W}'$ with $\{z, h_1(z)\} \subset c(f)^{-1}(V)$. So $y \in V$ from which it follows that $\{z, h_1(z)\} \subset c(f)^{-1}(\mathrm{St}(y, \mathcal{W}'))$. That's it.

We shall construct a continuous function $b : M \to (0, \frac{1}{4}]$ such that

(16) $(M \times_b I) \cap h_1(c(f)^{-1}(Y)) = \emptyset$ (see figure 2.8).

Recall that the definition of the variable product is in the section 2.4. Denote $Y' = h_1(c(f)^{-1}(Y))$. Take any open cover $\mathcal{G}$ of M. We may assume that it is countable and by the lemma 1.10 also that it is star-finite. Moreover, assume that all the sets in $\mathcal{G}$ have compact closure. Take any $U \in \mathcal{G}$. Since $\overline{U}$ has compact closure, there exists a number $b_U^* \in (0, \frac{1}{4})$ such that $(\overline{U} \times [0, b_U^*]) \cap Y' = \emptyset$. Construct modified Urysohn function $b_U : M \to [b_U^*, \frac{1}{4}]$ such that $b_U \restriction \overline{U} \equiv b_U^*$ and $b_U \restriction M - \mathrm{St}(U, \mathcal{G}) \equiv \frac{1}{4}$. Now define $b : M \to (0, \frac{1}{4}]$ by letting $b(m) = \min_{U \in \mathcal{G}} b_U(m)$ for every $m \in M$. Since $\mathcal{G}$ is star-finite, for each $m \in M$ only finitely many b_U's are less than $\frac{1}{4}$, so the minimum clearly exists and well-defines a continuous function. Moreover, $b(m) \leq b_U^*$ for each $m \in M$ and $U \in \mathcal{G}$ with $m \in U$. The condition (16) follows.

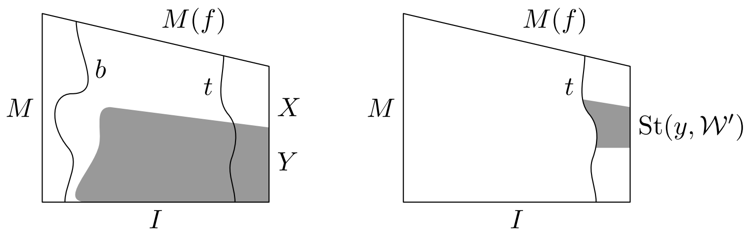


Figure 2.8: The mappings b and t.

Similarly, we shall construct a continuous function $t : M \to [\frac{3}{4}, 1)$ such that

(17) for every $y \in Y$ there is $V \in \mathcal{V}$ such that

$$\mathrm{St}(y, \mathcal{W}') \cup \left(f^{-1}(\mathrm{St}(y, \mathcal{W}')) \times_t' I\right) \subset V.$$

(See figure 2.8).

In the previous formula, the symbol $\times_t'$ is to mean the variable product, but in a sense that $X \times_t' I = \bigcup_{x \in X} (\{x\} \times [t(x), 1]) \subset X \times I$ for any space X and any continuous function $t : X \to I$. For every $y \in Y$ there is some $V \in \mathcal{V}$ with $\mathrm{St}(y, \mathcal{W}') \subset V$, because $\mathcal{W}'$ is star-refinement of $\mathcal{V}$. Since $\overline{\mathrm{St}(y, \mathcal{W}')}$ is a union of finitely many compact sets, it is also compact. Therefore there exists a number $t_y^* \in (\frac{3}{4}, 1)$ with

$$\mathrm{St}(y, \mathcal{W}') \cup \left(f^{-1}(\mathrm{St}(y, \mathcal{W}')) \times [t_y^*, 1)\right) \subset V.$$

Notice that there are only countably many sets of the type $\mathrm{St}(y, \mathcal{W}')$. We proceed in a same way as in the construction of b, just taking max instead of min. At the end we obtain a continuous $t : M \to [\frac{3}{4}, 1)$ such that $t(y) \geq t_y^*$. It is now clear that this t is as required.

Since $c(f)$ is proper map, the set $A = c(f)(h_1(c(f)^{-1}(Y)))$ is closed. By (12), $A \subset O$. Let $\beta : M \to I$ be an Urysohn function such that $\beta \restriction f^{-1}(A) \equiv 1$ and $\beta \restriction (M - f^{-1}(O)) \equiv 0$.

For every pair $(b, t) \in (0, \frac{1}{2}) \times (\frac{1}{2}, 1)$ we shall construct an isotopy $\Psi^{(b,t)} : I \times I \to I$ which has the following properties:

(18) $\Psi_0^{(b,t)} = 1_I$, $\Psi^{(b,t)}(b) = t$,

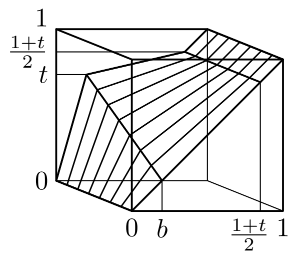


Figure 2.9: The graph of $\Psi^{(b,t)}$.

(19) $\Psi_i^{(b,t)} \restriction [\frac{1}{2}(1+t), 1] = 1_{[\frac{1}{2}(1+t),1]}$ for every $i \in I$.

(See the figure 2.9.)

We give the exact formula for this isotopy:

$$\Psi^{(b,t)}(x,i) = \begin{cases} \frac{t-b}{b} xi + x, & \text{for } (x,i) \in [0,b] \times I, \\ \left(1 + \frac{2(t-b)i}{2b-1-t}\right) x + \frac{(1+t)(b-t)i}{2b-1-t}, & \text{for } (x,i) \in [b, \frac{1}{2}(1+t)] \times I, \\ x, & \text{for } (x,i) \in [\frac{1}{2}(1+t), 1] \times I. \end{cases}$$

It is clear from the construction of $\Psi^{(b,t)}$ that this mapping depend continuously on b and t, i.e. the mapping $\Psi : I \times I \times (0, \frac{1}{2}) \times (\frac{1}{2}, 1) \to I$ defined by

$$\Psi \restriction I \times I \times \{b\} \times \{t\} = \Psi^{(b,t)} \quad \text{for } (b,t) \in (0, \tfrac{1}{2}) \times (\tfrac{1}{2}, 1)$$

is continuous. Now define the mapping $\Phi : M(f) \to M(f)$ by (see figure 2.10)

$$\begin{cases} \Phi(x) = x, & \text{for } x \in X, \\ \Phi(p,i) = (p, \Psi(i, \beta(p), b(p), t(p))), & \text{for } (p,i) \in M \times [0,1). \end{cases}$$

An easy check shows that Φ is a homeomorphism.

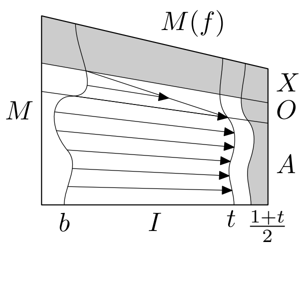


Figure 2.10: The mapping Φ.

We show

(20) if $y \in Y$ then there exists $V \in \mathcal{V}$ with $\Phi h_1(c(f)^{-1}(y)) \subset V$.

Choose an arbitrary $y \in Y$. By (15), $h_1(c(f)^{-1}(y)) \subset c(f)^{-1}(\mathrm{St}(y, \mathcal{W}'))$. From the definition of Φ it follows that it maps $c(f)^{-1}(\mathrm{St}(y, \mathcal{W}'))$ onto itself. Since β is 1 on $f^{-1}(A)$, by (18) and the definition of Φ we find that

(21) $\Phi h_1(c(f)^{-1}(y)) \subset \mathrm{St}(y, \mathcal{W}') \cup (f^{-1}(\mathrm{St}(y, \mathcal{W}')) \times'_t I)$.

By (17), the set on the right-hand side is contained in some $V \in \mathcal{V}$. That's it.

We define $h : M(f) \to M(f)$ by $h = h_1^{-1} \circ \Phi \circ h_1$. We claim that this h satisfies (3), (4) and (5). That $h \restriction X = 1_X$ is clear since $h_1 \restriction X = \Phi_X = 1_X$ by (13) and the definition of Φ, so (4) holds. Since $t(m) > \frac{1}{2}$ for every $m \in M$, h_1 is identity on $\Phi h_1(c(f)^{-1}(Y))$ (by (13) and (21)). So by (20), h satisfies (5). It therefore remains to check (3).

Take any $x \in M(f)$. If $h_1(x) \notin c(f)^{-1}(O)$, then it's clear by the construction of Φ that $\Phi h_1(x) = h_1(x)$, therefore $h(x) = x$ and we're done. More difficult case is when $h_1(x) \in c(f)^{-1}(O)$. By the definition of O we have $h_1(x) \in c(f)^{-1}(\mathrm{St}(y, \mathcal{W}'))$ for some $y \in Y$. By (12) there exists $V_0 \in \mathcal{W}'$ such that $\{x, h_1(x)\} \subset c(f)^{-1}(V_0)$. Therefore, $c(f)h_1(x) \in \mathrm{St}(y, \mathcal{W}') \cap V_0$, so there exists $V \in \mathcal{W}'$ such that $c(f)h_1(x) \in V \cap V_0$ and $y \in V$. By (12) there is $V_1 \in \mathcal{W}'$ with $\{\Phi h_1(x), h_1^{-1}\Phi h_1(x)\} \subset c(f)^{-1}(V_1)$. By the definition of Φ we have $\Phi h_1(x) \in c(f)^{-1}c(f)h_1(x)$. Therefore, $c(f)h_1(x) \in V_1$. It follows that the intersection $V \cap V_0 \cap V_1$ is nonempty, since it contains $c(f)h_1(x)$. Recall that $x \in c(f)^{-1}(V_0)$ and $h(x) \in c(f)^{-1}(V_1)$. Consequently, $\{x, h(x)\} \in c(f)^{-1}(\mathrm{St}(V, \mathcal{W}'))$. Since $\mathcal{W}' \overset{*}{<} \mathcal{W}$, there exists $W \in \mathcal{W}$ with $\mathrm{St}(V, \mathcal{W}') \subset W$. Moreover, $\mathrm{St}(V, \mathcal{W}') \cap Y \neq \emptyset$, since it both sets contain y. From this fact, it follows that W is of the form W_z for some $z \in Y$. By the choice of W_z there exists $U \in \mathcal{U}$ such that $c(f)^{-1}(W_z) \subset q^{-1}(U)$. Consequently, $\{x, h(x)\} \subset c(f)^{-1}(\mathrm{St}(V, \mathcal{W}')) \subset c(f)^{-1}(W_z) \subset q^{-1}(U)$. We are done, since this means that q and $q \circ h$ are $\mathcal{U}$-close. ♢

The continuation of the proof of the proposition 2.26. We shall now produce desired near homeomorphism $g : M(f) \to M(f)$. We just proved that q is near homeomorphism and g shall have the form $\gamma^{-1} \circ q$, where $\gamma : M(f) \to Z$ is carefully chosen homeomorphism approximating q.

From the proof of Bing Shrinking Criterion 1.13 it follows that q can be approximated by homeomorphisms of the form

$$\lim_{n \to \infty} q \circ f_n^{-1} \circ \cdots \circ f_1^{-1},$$

where each $f_n : M(f) \to M(f)$ is shrinking homeomorphism of q. By (4) we can choose each f_n to be identity on X. We conclude that there is a homeomorphism $\gamma : M(f) \to Z$ such that

(22) q and γ are as close as we wish,

(23) for every $x \in X$, $q(x) = \gamma(x)$.

Choose our γ to be $q_1^{-1}(\mathcal{F})$-close to q. As announced, we put $g = \gamma^{-1} \circ q$. The condition (2) clearly hold. We finish the proof by checking that also (1) hold. Since

$$(1_Z, q\gamma^{-1})_{q_1^{-1}(\mathcal{F})} \implies (q_1, q_1 q\gamma^{-1})_{\mathcal{F}} \implies (q_1 q, q_1 q\gamma^{-1}q)_{\mathcal{F}} \iff (c(f), c(f)\gamma^{-1}q)_{\mathcal{F}},$$

it's enough to show the first statement. But for any $z \in Z$, $\gamma^{-1}(z) \in M(f)$, so by choice of γ we have $\{z, q\gamma^{-1}(z)\} \subset q_1^{-1}(U)$ for some $U \in \mathcal{F}$. We are done. □

Proposition 2.28 (7.5.5*). *Let M be a $\mathcal{Q}$-manifold, let X be an ANR with the disjoint cells property, and let $g : M \to X$ be cell-like. If $proj : M \times \mathcal{Q} \to M$ denotes the projection, then the composition*

$$M \times \mathcal{Q} \xrightarrow{proj} M \xrightarrow{g} X$$

is a near homeomorphism. Consequently, X is homeomorphic to the $\mathcal{Q}$-manifold $M \times \mathcal{Q}$.

Proof. Put $N = M \times \mathcal{Q}$ and $f = g \circ proj$. Then N is a $\mathcal{Q}$-manifold and f is cell-like by the corollary 2.6. Let $p : N \times I \to N$ be the projection. Then the diagram

$$\begin{array}{ccc} N \times I & \xrightarrow{\pi_f} & M(f) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle c(f)} \\ N & \xrightarrow{f} & X \end{array}$$

clearly commutes. An easy diagram chase establishes that f is a near homeomorphism if and only if $c(f)$ is a near homeomorphism. We shall prove that $c(f)$ is a near homeomorphism.

Choose an open cover $\mathcal{U}$ of $M(f)$ and an open cover $\mathcal{V}$ of X. We shall construct a near homeomorphism $h : M(f) \to M(f)$ such that

(1) $c(f)$ and $c(f) \circ h$ are $\mathcal{V}$-close,

(2) for every $x \in X$ there exists $U \in \mathcal{U}$ with $h(c(f)^{-1}(x)) \subset U$.

It is clear that each homeomorphism $\phi : M(f) \to M(f)$ closely approximating h satisfies the conditions of Bing Shrinking Criterion 1.13. Therefore, we can conclude that then $c(f)$ is a near homeomorphism. (Recall that $c(f)$ is proper.)

The near homeomorphism h will be of the form $\tau \circ \mu^{-1} \circ g$, where g is a near homeomorphism such as in the proposition 2.26, and τ and μ are certain auxiliary homeomorphisms to be constructed below.

For the needs of this proof we recall and extend the definition of variable product. For the space X and continuous functions $t : X \to (0, 1]$ and $s : X \to [0, 1)$ we put

$$X \times_t I = \{\{x\} \times [0, t(x)] \mid x \in X\} \subset X \times I,$$
$$X \times_t' I = \{\{x\} \times [t(x), 1] \mid x \in X\} \subset X \times I,$$
$$X \times_s^t I = \{\{x\} \times (s(x), t(x)) \mid x \in X\} \subset X \times I.$$

Denote by $\mathcal{U} \cap X$ the cover $\{U \cap X \mid U \in \mathcal{U}\}$. Let $\mathcal{V}_1$ be an open cover of X such that all the sets in it have compact closure and $\mathcal{V}_1 \overset{*}{<} \mathcal{U} \cap X$. Let $\mathcal{V}_2$ an open cover of X such that $\mathcal{V}_2 \overset{*}{<} \mathcal{V}_1$ and $\mathcal{V}_2 \overset{*}{<} \mathcal{V}$. It is clear that for every $V \in \mathcal{V}_2$ there is $\lambda_V \in (0,1]$ such that there exists $U \in \mathcal{U}$ with the property that

$$c(f)^{-1}(\overline{V}) - N \times [0, \lambda_V] \subset U.$$

By the construction similar the that one in the previous proof we get a continuous function $\lambda : N \to (0,1]$ such that for each $V \in \mathcal{V}_2$ there exists $U \in \mathcal{U}$ with

$$(c(f)^{-1}(V)) - (N \times_\lambda I) \subset U.$$

(See the figure 2.11.)

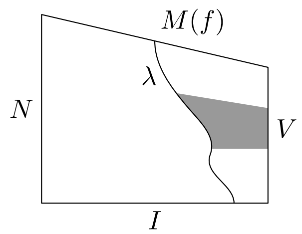


Figure 2.11: The set $c(f)^{-1}(V) - (N \times_\lambda I)$

Claim 1. There exist an open cover $\mathcal{W}$ of N and a sequence of continuous functions $r_i : N \to (0,1]$ such that

(3) $r_i(x) < r_{i+1}(x)$ for all $i \in \mathbb{N}$ and $x \in N$,

(4) for every $x \in N$ there exists $n \in \mathbb{N}$ such that $r_n(x) \geq \lambda(x)$,

(5) for every $W \in \mathcal{W}$ and $n \in \mathbb{N}$ there exists $U \in \mathcal{U}$ such that

$$W \times_{r_{n-2}}^{r_{n+2}} I \subset U,$$

if we define $r_{-1}(x) = r_0(x) = 0$ for every $x \in N$,

(6) λ intersects at most one of the r_j's on each set of the form $W \times [0,1)$, $W \in \mathcal{W}$.

Proof of the claim. Write $N = \bigcup_{i \in \mathbb{N}} C_i$, where $C_1 \subset \operatorname{int} C_2 \subset C_2 \subset \cdots$ are compacta. Let d_N be any metric on N, let d_I denote the standard metric on I and let d' denote the metric on $N \times [0,1)$ such that $d'((x_1,y_1),(x_2,y_2)) = d_N(x_1,x_2) + d_I(y_1,y_2)$. For every $i \in \mathbb{N}$, the cover "$\mathcal{U} \cap (C_{2i+1} \times_{\frac{1+\lambda}{2}} I)$" has a Lebesgue number, say δ_i, with respect to this metric, since it is an open cover of the compact set. We may assume that $\delta_1 > \delta_2 > \cdots$.

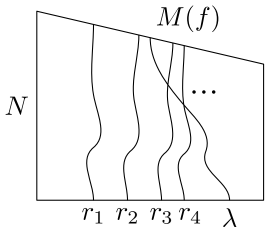


Figure 2.12: The functions r_i

For each $i \in \mathbb{N}$, let $u_i : N \to [0,1]$ be Urysohn function such that $u_i \restriction C_{2i-1} = 1$ and $u_i \restriction N - \operatorname{int}(C_{2i}) = 0$.

First, we inductively construct the sequence $(r'_i)_{i\in\mathbb{N}}$ of continuous functions from N to $(0,\infty)$ such that

(7) $r'_i(x) < r'_{i+1}(x)$ for every $x \in N$ and $i \in \{0,1,2,\dots\}$,

(8) for any $i \in \mathbb{N}$ and $x \in C_{2i+1}$ we have $r'_{i+2}(x) - r'_{i-2}(x) < \delta_i$,

(9) for every $x \in N$ there exists $n \in \mathbb{N}$ such that $r'_n(x) \geq \lambda(x)$,

if we put $r'_{-1}(x) = r'_0(x) = 0$ for every $x \in N$.

We perform the inductive step in such a way that we maintain the condition (7) and the condition

(10) if $x \in C_{2i+1} - \operatorname{int}(C_{2i-2})$ for some $i \in \mathbb{N}$, then $\frac{\delta_{i+1}}{5} < r'_j(x) - r'_{j-1}(x) < \frac{\delta_i}{5}$ for each $j \in \mathbb{N}$.

It is easy to see that the conditions (8) and (9) follow from this one.

As the first step of the induction we define $r'_1 : N \to (0,\infty)$. For every $i \in \mathbb{N}$, for each $x \in C_{2i+1} - \operatorname{int}(C_{2i-2})$ we put $r'_1(x) = \frac{\delta_{i+1}}{5} + u_i(x)(\frac{\delta_i}{5} - \frac{\delta_{i+1}}{5})$. It's clear that the functions are defined continuously on the closed sets covering N such that the definitions agree on intersections. Therefore, the function r'_1 is well-defined, continuous and satisfies (7) and (10).

Now suppose we have already defined all r'_j's for $j \leq n$. We put $r'_{n+1}(x) = r'_n(x) + \frac{\delta_{i+1}}{5} + u_i(x)(\frac{\delta_i}{5} - \frac{\delta_{i+1}}{5})$ if $x \in C_{2i+1} - \operatorname{int}(C_{2i-2})$ for some $i \in \mathbb{N}$. Clearly, function defined in this way suffices for our needs.

Now just define $r_j(x) = \min\big(\lambda(x) + (1 - \frac{1}{2(j+1)})(\min(\delta_i, \frac{1-\lambda(x)}{2})), r'_j(x)\big)$ for $j \in \mathbb{N}$ and $x \in C_{2i+1}$. An easy check shows that the conditions analogous to (7)–(9) for r_j's are satisfied, and r_j's are continuous functions into $(0,1)$. So also (4) and (3) hold.

What remains is to construct the cover $\mathcal{W}$. Pick $x \in \operatorname{int}(C_{2i+1})$, $i \in \mathbb{N}$. By (9) there is $n \in \mathbb{N}$ with $r_n(x) \geq \lambda(x)$. By (8), each of the sets $\{x\} \times [r_{j-2}(x), r_{j+2}(x)]$ for $j \leq n$ has d'-diameter less than δ_i. Moreover, also the set $\{x\}\times[r_n(x), \lambda(x)+\delta_i]$ has d'-diameter less than δ_i and $r_n(x) < r_m(x) < \lambda(x)+\delta_i$ for all natural numbers $m \leq n$. Now by continuity of all involved functions and by the fact that there are only finitely many sets of the type we just described, there is a neighborhood W_x of x in N such that the all the sets $W_x \times_{r_{j-2}}^{r_{j+2}} I$ have d'-diameter less than δ_i and $W_x \subset \mathrm{int}(C_{2i+1})$. Therefore, each such set is contained in some $U \in \mathcal{U}$. This fixes the condition (5).

Let $\mathcal{W} = \{W_x \mid x \in X\}$. We finish the proof of the claim by remark that we can (by possibly taking smaller sets) adjust $\mathcal{W}$ in such a way that satisfies (6). It is clear from the construction that also the rest of the conditions in the formulation of the claim hold. $\diamondsuit$

Now take a metric d on $M(f)$ such that the cover of all open 2-d-balls refine the cover

$$\{W \times_{r_{n-1}}^{r_{n+1}} I \mid n \in \mathbb{N}, W \in \mathcal{W}\}.$$

Throughout the rest of the proof of the proposition we denote the diameter of the set $B \subset M(f)$ with respect to the metric d as $\mathrm{diam}_d(B)$. Note that for the set $B \subset M(f)$,

(11) $\mathrm{diam}_d(B) < 1$ and $B \cap (N \times_\lambda I) \neq \emptyset$ implies that there exist $W \in \mathcal{W}$ and $n \in \mathbb{N}$ such that $B \subset W \times_{r_{n-1}}^{r_{n+1}} I$.

By the assumption, X is the locally compact ANR with the disjoint cells property, so by the theorem 1.9 and the theorem 2.14 there exists a proper homotopy $H : N \times I \to X$ limited by $\mathcal{V}_2$ such that $H_0 = f$ and H_1 is Z-embedding. Define $F : N \times I \to M(f)$ by

$$F(x,t) = \begin{cases} \pi_f(x, 2t) & \text{for } (x,t) \in N \times [0, \frac{1}{2}] \\ H(x, 2t-1) & \text{for } (x,t) \in N \times [\frac{1}{2}, 1]. \end{cases}$$

F is clearly a proper homotopy, F_0 is an inclusion, $\xi = F_1 : N \to X \subset M(f)$ is an Z-embedding. Moreover, F is limited by $c(f)^{-1}(\mathcal{V}_2)$.

By the corollary 2.25, $M(f)$ is $\mathcal{Q}$-manifold. Furthermore, $N \times \{1\}$ is a Z-set in $N \times I$, so by the proposition 2.10 we have that $X \in \mathcal{Z}(M(f))$. Therefore, also $\xi(N) \subset X$ is a Z-set in $M(f)$. In addition, $N \times \{0\} \in \mathcal{Z}(M(f))$. By the theorem 2.22 there exists an isotopy $G : M(f) \times I \to M(f)$ such that

(12) G is limited by $c(f)^{-1}(\mathcal{V}_2)$,

(13) $G_0 = 1_{M(f)}$ and $G_1 \restriction N = \xi$.

Put $\mu = G_1 : M(f) \to M(f)$. It follows that $\mu \restriction N = \xi$, so $\mu(N) = \xi(N)$. Moreover,

(14) $c(f) \circ \mu$ and $c(f)$ are $\mathcal{V}_2$-close.

We have that $\xi(N) \in \mathcal{Z}(X)$, so by the proposition 2.26 there is a near homeomorphism $g : M(f) \to M(f)$ such that

(15) $c(f) \circ g$ and $c(f)$ are $\mathcal{V}_2$-close,

(16) $g \restriction g^{-1}(\xi(N)) = c(f) \restriction c(f)^{-1}(\xi(N))$.

Claim 2. For every $x \in X$, if $\mu^{-1}g(c(f)^{-1}(x)) \cap N \neq \emptyset$, then $\mu^{-1}g(c(f)^{-1}(x))$ is a single point.

Proof of the claim. From the assumption follows that $g(c(f)^{-1}(x)) \cap \xi(N) \neq \emptyset$, so $c(f)^{-1}(x) \cap g^{-1}\xi(N) \neq \emptyset$. By (16) we have $x \in \xi(N)$. Again by (16), $g(c(f)^{-1}(x))$ is a single point. But since μ is a homeomorphism, also $\mu^{-1}g(c(f)^{-1}(x))$ is a single point. ♢

Define

$$A = \{\mu^{-1}g(c(f)^{-1}(x)) \mid x \in X \text{ and } \operatorname{diam}_d\left(\mu^{-1}g(c(f)^{-1}(x))\right) \geq 1\}.$$

It is union of locally finite system of closed sets, so it is itself closed subset of $M(f)$. By claim 2, $A \cap N = \emptyset$. So by the construction summoned a few times before, there exists a continuous function $s_1 : N \to (0,1)$ such that $A \cap (N \times_{s_1} I) = \emptyset$. We may clearly assume that $s_1(x) < r_1(x)$ for all $x \in N$. Put

$$A' = \{\mu^{-1}g(c(f)^{-1}(x)) \mid x \in X \text{ and } \mu^{-1}g(c(f)^{-1}(x)) \cap \pi_f(N \times'_{s_1} I) \neq \emptyset\}.$$

This set is again closed and has empty intersection with N, so there exists a continuous function $s_2 : N \to (0,1)$ such that $A' \cap (N \times_{s_2} I) = \emptyset$. Clearly $s_2(x) < s_1(x)$ for every $x \in N$. We repeat this construction inductively to get a sequence $(s_n)_{n\in\mathbb{N}}$ of continuous functions from N to $(0,1)$ with the properties:

(17) if for some $x \in N$ $\operatorname{diam}_d(\mu^{-1}g(c(f)^{-1}(x))) \geq 1$ then $\mu^{-1}g(c(f)^{-1}(x)) \cap (N \times_{s_1} I) = \emptyset$,

(18) each $\mu^{-1}g(c(f)^{-1}(x))$ intersects at most one of the "levels" $\{(x, s_i(x)) \mid x \in N\} \subset N \times I$,

(19) for every $x \in N$ and every $i \in \mathbb{N}$ we have $s_i(x) > s_{i+1}(x)$.

Now we need a homeomorphism $\tau : M(f) \to M(f)$ which performs "a push" in the I-coordinate (we think of $M(f)$ as a quotient space of $N \times I$). It means that the following condition holds:

(20) $c(f)(x) = c(f)(\tau(x))$ for every $x \in M(f)$.

Moreover, it should "shrink" the fibres $\mu^{-1}g(c(f)^{-1}(x))$ which are "big" (i.e. the set A is "pushed to the right behind λ"; see the figure 2.13) and not "expand" the rest very much. We can achieve this goal using r_i's and s_i's.

We need a sequence $(t_i)_{i\in\mathbb{N}}$ of continuous functions from N to $(0,1)$ such that

(21) for every $x \in N$ there are only finitely many n's such that $s_n(x) \neq t_n(x)$,

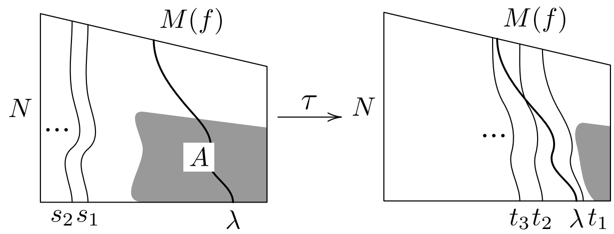


Figure 2.13: The homeomorphism τ

(22) $t_1(x) \geq \lambda(x)$ and $t_1(x) > s_1(x)$ for every $x \in N$,

(23) for every $i \in \mathbb{N}$ and every $x \in N$ we have $t_i(x) > t_{i+1}(x)$,

(24) for each $W \in \mathcal{W}$ and each $i \in \mathbb{N}$ we have $W \times_{t_{i-1}}^{t_{i+1}} I \subset W \times_{r_{j-2}}^{r_{j+2}} I \subset U \in \mathcal{U}$ for some $j \in \mathbb{N}$ and $U \in \mathcal{U}$.

Claim 3. Such a sequence of t_i's exists.

Proof of the claim. We use the sequence we already have, namely r_i's. Let $E_j = \{x \in N \mid r_{i-1}(x) \leq \lambda(x) \leq r_j(x)$ for $j \in \mathbb{N}$ and put $F_i = \bigcup_{1 \leq j \leq i} E_i$. (Once again, we regard $r_0 \equiv 0$.)

By (4) there exists $n \in \mathbb{N}$ with $E_n \neq \emptyset$. If for some $m \in \mathbb{N}$ greater than n we have $E_m = \emptyset$, then by the continuity of λ we have $E_k = \emptyset$ for all $k \geq m$ and we can define $t_i = r_{m-i}$ for $i = 1, 2, \ldots m-1$ and $t_i = s_{i-m+1}$ for $i = m, m+1, \ldots$. An easy check show that this is enough.

So suppose that $E_m \neq \emptyset$ for all $m \geq n$. Note that by (6) each $W \in \mathcal{W}$ intersects at most two (consecutive) E_i's. In other words,

(25) $\mathrm{St}(F_i, \mathcal{W}) \subset F_{i+1}$ for all $i \in \mathbb{N}$.

We construct required functions inductively. First we define all of them on F_n and then inductively extend their domain over F_i's.

As a first step, for $x \in F_n$ put $t_i(x) = r_{n+2-i}(x)$ for $i = 1, 2, \ldots, n+1$ and $t_i(x) = s_{i-n-1}(x)$ for $i = n+2, n+3, \ldots$ The required conditions clearly hold.

To perform the step of induction, all we have to care about is to extend already defined functions continuously and retain the conditions (21)–(24). So suppose that our t_i's are defined on F_m for some $m \geq n$. By (25) we have that $\mathrm{St}(F_m, \mathcal{W}) \subset F_{m+1}$. There exist a closed set D_1 and an open set D_2 such that

$$F_m \subset \mathrm{int}(D_1) \subset D_1 \subset D_2 \subset \overline{D_2} \subset \mathrm{St}(F_m, \mathcal{W}) \subset F_{m+1}.$$

Let $v : N \to I$ be Urysohn function such that $v \restriction D_1 \equiv 0$ and $v \restriction N - D_2 \equiv 1$. Now for $x \in F_{m+1} - \mathrm{int}(F_m)$ define

$$t_i(x) = \begin{cases} r_{m+2-i}(x) + v(x)(r_{m+3-i}(x) - r_{m+2-i}(x)) & \text{for } 1 \leq i \leq m+1 \\ s_{m+1-n}(x) + v(x)(r_1(x) - s_{m+1-n}(x)) & \text{for } i = m+2 \\ s_{i-n-1}(x) & \text{for } i \geq m+3. \end{cases}$$

We are done. ♢

We construct τ. Fix $x \in N$. Let $n(x)$ denote such natural number that for all natural $i \geq n(x)$ we have $s_i(x) = t_i(x)$. Define $\tau'_x : [0,1) \to [0,1)$ by the statement that it takes $[0, s_{n(x)}(x)]$ linearly onto $[0, t_{n(x)}(x)]$, it takes $[s_{i+1}(x), s_i(x)]$ linearly onto $[t_{i+1}(x), t_i(x)]$, it takes $[s_1(x), \frac{1}{2}(t_1(x)+1)]$ linearly onto $[t_1(x), \frac{1}{2}(t_1(x)+1)]$ and finally it is identity on $[\frac{1}{2}(t_1(x)+1), 1)$. It is clearly a homeomorphism of $[0,1)$ for each $x \in N$.

Let the mapping $\tau' : N \times [0,1) \to N \times [0,1)$ be defined by $\tau'(x,t) = (x, \tau'_x(t))$. Since all t_i's and s_i's are continuous, τ' is continuous. It is clear that it is also a homeomorphism which is the identity on $N \times'_{\frac{1}{2}(t_1(x)+1)} [0,1)$. Consequently, if we define $\tau : M(f) \to M(f)$ as τ' on $N \times [0,1)$ and the identity on the rest, it is also a homeomorphism. It is easy to see that it has all the required properties, i.e. the condition (20) is true, $\tau(W \times_{s_{i-1}}^{s_{i+1}} I) \subseteq W \times_{t_{i-1}}^{t_{i+1}} I$ for each $W \in \mathcal{W}$ and $i \in \mathbb{N}$ and $\tau(A) \subset M(f) - (N \times_\lambda I)$.

Claim 4. For every $x \in X$, $\tau\mu^{-1}g(c(f)^{-1}(x)) \subset U$ for some $U \in \mathcal{U}$.

Proof of the claim. Take any $p \in X$. By (15), $g(c(f)^{-1}(p)) \subset c(f)^{-1}(V)$ for some $V \in \mathcal{V}_2$ containing p. Now it follows from (14) that

(26) for any $p \in X$ there exists $V_p \in \mathcal{V}_2$ containing p such that $\mu^{-1}g(c(f)^{-1}(p)) \subset c(f)^{-1}(\mathrm{St}(V_p, \mathcal{V}_2))$.

Take any $x \in X$ and assume that $\mu^{-1}g(c(f)^{-1}(x))$ is not a single point. (If it were, the claim clearly holds.)

Case 1: $\mu^{-1}g(c(f)^{-1}(x)) \cap (N \times_{s_2} I) = \emptyset$. Then by the definition of τ and (26) we have that

$$\tau\mu^{-1}g(c(f)^{-1}(x)) \subset \bigcup\{c(f)^{-1}(V') - (N \times_\lambda I) \mid V' \in \mathcal{V}_2 \text{ and } V' \cap V_x \neq \emptyset\}$$
$$\subset c(f)^{-1}(\mathrm{St}(V_p, \mathcal{V}_2)) - (N \times_\lambda I) \subset U,$$

for some $U \in \mathcal{U}$.

Case 2: $\mu^{-1}g(c(f)^{-1}(x)) \cap (N \times_{s_2} I) \neq \emptyset$. By (17), $\mathrm{diam}_d\,(\mu^{-1}g(c(f)^{-1}(x))) < 1$, so by (18) and claim 2 there exists $i \in \mathbb{N}$ such that

$$\mu^{-1}g(c(f)^{-1}(x)) \subset N \times_{s_{i-1}}^{s_{i+1}} I.$$

By (11) there exists $W \in \mathcal{W}$ such that

$$\mu^{-1}g(c(f)^{-1}(x)) \subset W \times I.$$

Putting these facts together and using the definition of τ we have

$$\tau\mu^{-1}g(c(f)^{-1}(x)) \subset W \times_{t_{i-1}}^{t_{i+1}} I$$

and by (24) the last set is contained in some $U \in \mathcal{U}$. ♢

So far we proved that we picked up τ, g and μ such that the condition (2) is satisfied. We finish the proof by showing that $c(f)\tau\mu^{-1}g$ and $c(f)$ are $\mathcal{V}$-close — the condition (1). Take any $x \in M(f)$, put $y = c(f)(x) \in X$. By (26) there is $V \in \mathcal{V}_2$ containing y such that

$$\mu^{-1}g(c(f)^{-1}(y)) \subset c(f)^{-1}(\mathrm{St}(V, \mathcal{V}_2)).$$

By the definition of τ we get that also

$$\tau\mu^{-1}g(c(f)^{-1}(y)) \subset c(f)^{-1}(\mathrm{St}(V, \mathcal{V}_2)),$$

and finally by the definition of $\mathcal{V}_2$ the last set is contained is some $V \in \mathcal{V}$. We are done. □

Theorem 2.29 (Stability of $\mathcal{Q}$-manifolds; 7.5.6*). *If M is a $\mathcal{Q}$-manifold, then $M \times \mathcal{Q}$ and M are homeomorphic. Moreover, the projection $proj : M \times \mathcal{Q} \to M$ is a near-homeomorphism.*

Proof. By the theorem 2.15, M is the locally compact ANR with the disjoint cells property. The identity $1_M : M \to M$ is obviously cell-like. So by the proposition 2.28 we conclude that the composition

$$M \times \mathcal{Q} \xrightarrow{proj} M \xrightarrow{1_M} M$$

is a near homeomorphism. We are done. □

Theorem 2.30 (Toruńczyk's Approximation Theorem; 7.5.7*). *Let M be a $\mathcal{Q}$-manifold, let X be a locally compact ANR and let $f : M \to X$ be cell-like. Then the following statements are equivalent:*

(1) f is a near homeomorphism,

(2) X has the disjoint-cells property.

Proof. The implication (1) $\Longrightarrow$ (2) is clear, since every $\mathcal{Q}$-manifold has the disjoint cells property. By the proposition 2.28, the composition

$$M \times \mathcal{Q} \xrightarrow{proj} M \xrightarrow{f} X$$

is a near homeomorphism. Since by the theorem 2.29 $proj : M \times \mathcal{Q} \to M$ is a near homeomorphism, it follows easily that also f is a near homeomorphism. □

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